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MODIFYING SOYSTER'S MODEL FOR THE SYMMETRIC TRAVELING SALESMAN PROBLEM WITH INTERVAL TRAVEL TIMES

Nayoung Cho, Samuel Burer and Ann Melissa Campbell

Department of Management Sciences
University of Iowa
Iowa City, IA, 52242-1994
U. S. A.
e-mail: ann-campbell@uiowa.edu

Abstract

In this paper, we examine a variant of the symmetric traveling salesman problem in which travel time uncertainty is modeled by interval ranges. We introduce a new model that incorporates some ideas from existing robust optimization models - most importantly, the ability to control the model's level of conservatism - but does so without increasing computation time. We discuss theoretical properties of this model and demonstrate its performance compared to other robust optimization approaches in a series of computational experiments.

1. Introduction

The classic traveling salesman problem (TSP) finds a tour visiting all customers exactly once and returning to the point of departure such that the overall travel time is minimized. The TSP is NP-hard and computationally quite challenging for realistic problem sizes. Beyond these difficulties, the

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classic TSP also ignores complexities that occur in real-world routing problems. One of the most notable is uncertain travel times between customers, and such uncertainties can significantly impact the actual travel time associated with a tour. This can be critical because if the actual travel time is quite a bit longer than planned, for example, customers may be left unserved or driver overtime may be required.

In this paper, we explore methods for incorporating travel-time uncertainty in the symmetric version of the TSP. We model travel-time uncertainty through intervals in accordance with the robust optimization literature (see Section 2). Specifically, for each edge e in the network, the uncertain travel time is modeled as the interval $[l_e, u_e]$, and we call this variant of the symmetric traveling salesman problem the iTSP. Our goal is to identify techniques that can handle the uncertainty well but also be computationally tractable.

Two well-known techniques in the robust optimization literature can be adapted to iTSP. The first, described fully in Subsection 3.1, is due to (Soyster [37]) and is a worst-case model. For iTSP, this translates into finding the tour that minimizes the sum of the u_e values of the edges in the tour. Computationally, Soyster's model is equivalent to solving a single deterministic TSP. However many researchers characterize the corresponding solution as too conservative or pessimistic (Ben-Tal and Nemirovski [5, 7]). In other words, Soyster's model prevents a tour from taking too long in the worst case (e.g. it may never require driver overtime), but by only considering the u_e values, the tour will likely be poor in most cases.

The second technique, which can be adapted to iTSP, is due to Bertsimas and Sim [10, 11]. Their research grew out of efforts (Ben-Tal and Nemirovski [5, 7]) to improve upon Soyster's model by allowing the user to control the level of conservatism. As far as we are aware, Bertsimas and Sim's approach is the only one in the robust optimization literature which controls conservatism and applies to combinatorial problems with uncertainty present solely in the objective, as is the case with iTSP. In contrast to Soyster, Bertsimas and Sim note that it is highly unlikely that the travel time on all edges will realize their worst-cases u_e simultaneously. Accordingly, the Bertsimas-Sim model requires a single parameter Γ and finds a minimum travel-time tour under the following definition: given a tour, its travel time is the maximum total travel time among all time realizations that have Γ edges at their worst-case u_e and the remaining edges at their best case l_e. Said differently, the travel time of a given tour is based on the l_e values plus a penalty that equals the worst possible delay caused by some set of Γ edges, and the Bertsimas-Sim model chooses the minimum-travel-time tour among all tours. In this sense, Bertsimas-Sim focuses its conservatism on just Γ edges, whereas Soyster's model is conservative on all edges.

Computationally, this model requires the solution of $|E|+1$ separate deterministic TSPs, where $|E|$ is the total number of edges in the network. For example, on a complete network with n customers, $n(n-1)/2+1$ TSP problems must be solved. This approach can be quite expensive computationally for practical problem sizes.

In this paper, we propose an alternative to both Soyster and Bertsimas-Sim. It is a logical extension of Soyster's model, which we refer to as modified-Soyster, that controls for conservatism and yet retains the same complexity computationally as solving a single deterministic TSP. In contrast to the focused conservatism of Bertsimas-Sim on Γ edges of a given tour x, modified-Soyster takes a more balanced view of the entire uncertainty of x. It considers both the u_e and l_e values of all edges. The model finds a minimum travel-time tour where the travel time of each edge e is given by $l_e + \gamma(u_e - l_e)$ for a parameter $\gamma \in [0, 1]$, which reflects the user's conservatism. This form of conservatism seems a natural fit for managers, since the choice of γ has a clear translation to the pessimism or optimism of a manager. For example, $\gamma = 0.25$ reflects that a manager only wants to put a 25% emphasis on the u_e values and thus has a more optimistic perspective. A γ of 0.75 reflects that a manager wants to put a 75% emphasis on the worst case and thus has a more pessimistic perspective. We detail this new model in Section 4 and prove several intriguing theoretical properties of the robust solutions it provides, which also have managerial consequences. In Section 5, we perform computational experiments to evaluate the comparative runtime of modified-Soyster and evaluate how the solutions differ from Bertsimas-Sim.

We also compare modified-Soyster computationally with a fourth model- a robust deviation model developed by Montemanni et al. [32], which is the only existing method we found in the literature designed specifically for iTSP. For a specified tour, the tour's maximum deviation is the difference of two values. The first value is the total time of the tour when travel times are realized as follows: all edges in the tour are set to their worst-case values u_e, while all remaining edges in the network are set to their best-case values l_e. The second value is the optimal tour time under the same realization. Among all tours, the method of Montemanni et al. selects a tour having minimum maximum deviation. This is also sometimes termed as minimizing maximum regret. Using such an approach, the authors avoid tours that only do well in the best or worst cases, but they also create a challenging new optimization problem which requires specialized algorithms. In order to improve computation, the authors also develop heuristics. More details are given in Subsection 3.3.

The paper is organized as follows. Section 2 reviews the related literature. In Section 3, we describe Soyster's model, the model of Bertsimas and Sim, and the robust deviation model of Montemanni et al. Section 4 describes our modification of Soyster's model and its theoretical properties. Section 5 details our computational experiments. Our experiments highlight the speed of modified-Soyster and demonstrate that the solutions created by modified-Soyster perform as well as the solutions for the other more time-consuming models. We also demonstrate certain characteristics of the solutions created by modified-Soyster that differ from the solutions created by Bertsimas-Sim. Managerial insights are provided in Section 6 along with a discussion of future work.

2. Literature Review

The TSP is one of the most intensively studied problems in combinatorial optimization. Many variants and generalizations of the problem have been studied over the years, starting from the pioneering work of Dantzig et al. [19]. Comprehensive surveys can be found in Lawler et al. [29], Reinelt [35], Gutin and Punnen [22] and Applegate et al. [2]. The TSPLIB is a well known library of TSP test instances (Reinelt [34]). The TSP is also the basis for many applications in transportation, such as the vehicle routing problem (VRP).

Most literature in transportation dealing with data uncertainty have focused on *stochastic demand*, where it is unknown whether any particular customer will need to be visited or, more generally, the amount of customer demand is uncertain. For the first type of stochastic demand, the probabilistic TSP (PTSP) incorporates demand uncertainty by assuming the need to visit a customer is probabilistic. An analytical framework for the PTSP can be found in Jaillet [23], and Laporte et al. [28] provide an exact algorithm. There have been several papers on heuristic approaches such as Bertsimas et al. [13], Bertsimas and Howell [9], Campbell [14] and Tang and Miller-Hooks [38]. Campbell and Thomas [15, 16] study a variant of the PTSP with deadlines to visit customers (PTSPD). For the second type of stochastic demand, the stochastic vehicle routing problem (SVRP) models demand at a customer as a random variable. In addition, the probabilistic VRP (PVRP) assumes both types of stochastic demand. Overviews of research in transportation problems with stochastic demand can be found in Powell et al. [33], Bertsimas and Simchi-Levi [12] and Gendreau et al. [21].

Several papers in transportation consider travel time uncertainty. For example, Laporte et al. [27] examine the VRP with stochastic travel times assuming that the service time at each customer is also stochastic. They provide two solution approaches. The first is a chance-constrained model which ensures the probability that the total time (including service times at the customers) exceeds a given time limit is less than a certain threshold. The second is a stochastic-programming-with-recourse model with expected penalty costs proportional to the excess time incurred beyond the given time limit. Algorithmically, they propose branch-and-cut algorithms and report computational results for moderately sized problems involving a limited number of random scenarios. Lambert et al. [26] consider the possibility of congestion, which makes the travel time between nodes uncertain, and incorporate a constant penalty in the objective for any tour exceeding a given time limit. The authors then propose a heuristic based on Clarke and Wright [17]. Kenyon and Morton [25] also propose a VRP model with stochastic travel times. Their model minimizes the longest tour (whereas the two papers described previously consider the total travel time for all tours). They present a branch-and-cut scheme embedded within a sampling-based procedure to solve the problem. Finally, Russell and Urban [36] investigate the VRP with stochastic travel times and soft time-window constraints and propose a tabu-search metaheuristic for its solution.

As mentioned in the Introduction, existing robust optimization techniques can be applied to iTSP. For network problems, robust techniques often assume only an interval estimate of edge length and that the actual edge length may be realized anywhere within the interval. More specifically, let $G = (V, E)$ be an undirected network. On each edge $e \in E$, the uncertain travel time is modeled as the interval $[l_e, u_e]$ with $0 \leq l_e \leq u_e$. No additional information about the travel times, e.g., a probability distribution, is assumed. (If such information is known, then stochastic programming models may be applicable.) Examples of the use of interval estimates can be found in Zieliński [39], Averbakh and Lebedev [4], Kasperski and Zieliński [24], Montemanni and Gambardella [30] and Montemanni et al. [31] for shortest path problems, Montemanni et al. [32] for TSPs, and Bertsimas and Sim [10] for various network flow problems.

Besides the well-known models mentioned in the Introduction and detailed in Section 3, there are two additional robust optimization models that control for conservatism. First, Ben-Tal and Nemirovski [5], Ben-Tal and Nemirovski [6, 7] and Ben-Tal et al. [8] propose ellipsoidal uncertainty sets that exclude the unlikely joint extreme realizations in Soyster's model.

This allows the degree of conservatism of the solution to be controlled by choosing ellipsoids of differing volumes. However, this approach transforms, for example, an uncertain linear program into a deterministic second-order cone program which requires significantly more computational effort compared to Soyster's linear model. Further, this approach is a challenge to apply directly to discrete robust optimization problems due to its nonlinearities, and thus, we do not apply this approach to iTSP in this paper. Second, Fischetti and Monaci [20] introduce the concept of "light robustness" which controls for conservatism but appears best suited for handling uncertainty in the constraints rather than the objective.

3. Background

In this section, we detail the methods that have motivated the modified-Soyster model (described fully in Section 4) and/or serve as sources of comparison in Section 5. We also present an example to illustrate how the models differ.

3.1. Soyster's model

Soyster [37] considers the modification of the linear optimization problem

$$\begin{aligned} &\text{minimize } \mathbf{c}^T\mathbf{x} \\ &\text{subject to } \mathbf{A}\mathbf{x} \le \mathbf{b} \end{aligned} \tag{1}$$

to consider uncertainty both in the constraint matrix and in the objective function. Suppose that $\mathbf{x} \geq 0$ is implied by $\mathbf{Ax} \leq \mathbf{b}$. When uncertainty occurs and is modeled only in the objective with $\tilde{\mathbf{c}}$ taking values in $c \pm \hat{c}$, Soyster shows that the corresponding robust formulation is

$$\begin{aligned} &\text{minimize } (\mathbf{c} + \hat{\mathbf{c}})^T\mathbf{x} \\ &\text{subject to } \mathbf{A}\mathbf{x} \le \mathbf{b}. \end{aligned} \tag{2}$$

In particular, note that because $\mathbf{x} \geq 0$, the downside realization $\mathbf{c} - \hat{\mathbf{c}}$ is irrelevant.

Even though Soyster developed his ideas for linear programs, they can also be applied directly to integer programs. For example, under Soyster's model, iTSP, which has uncertainty only in the objective and also has $\mathbf{x} \geq 0$, becomes

$$\min_{x \in \mathcal{TSP}} u^T x, \tag{3}$$

where

$$\mathcal{TSP} = \{x \in \mathbf{R}^{|E|} : x \text{ is an indicator vector of a tour}\}$$

and the u vector contains the u_e values for all edges e. Here, the vector of worst-case travel times u is identified with the upper-bound vector $c + \hat{c}$ of Soyster's method. This formulation in (3) is equivalent to a single deterministic TSP and thus can be solved using any TSP solver.

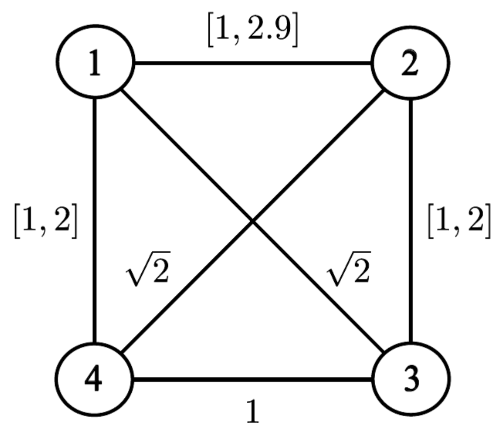

Figure 1. Example of a robust TSP problem with interval times.

Figure 1 depicts an example for $n = 4$. The uncertainty intervals are shown on the edges, except when $l_e = u_e$, in which case only the single value is shown. The network has three possible tours, $t_1 = (1, 2, 3, 4, 1)$, $t_2 = (1, 2, 4, 3, 1)$ and $t_3 = (1, 4, 2, 3, 1)$.

For Soyster's model, the time of each tour is as follows:

Tour	Robust time
t_1	$4 + 1.9 + 1 + 1 + 0 = 7.9$
t_2	$2 + 2\sqrt{2} + 1.9 + 0 + 0 + 0 = 3.9 + 2\sqrt{2}$
t_3	$2 + 2\sqrt{2} + 1 + 1 + 0 + 0 = 4 + 2\sqrt{2}$

So Soyster's model chooses t_2 as the optimal robust tour.

3.2. Bertsimas-Sim model

Bertsimas and Sim [10, 11] also consider the modification of (1) to consider uncertainty both in the constraints and the objective. In their models, they allow the user to control the level of conservatism of the robust optimal solution. Bertsimas and Sim [10] also show that their model is applicable to combinatorial problems such as iTSP. This requires a single parameter Γ and optimizes

$$\min_{\mathbf{x}\in \mathcal{TSP}} \left(l^T x + \max_{|S|=\Gamma} \left\{ \sum_{e\in S} (u_e - l_e) x_e \right\} \right) \tag{4}$$

with the l_e and u_e values defined as in the previous section. For the example in Figure 1, the optimal robust tours for different values of Γ are as follows:

Γ	Optimal robust tour
0	t_1
1	t_3
≥ 2	t_2

When $\Gamma = 0$, the model simply minimizes the optimistic bound $l^T x$ which yields t_1. However, depending on the value of Γ, t_2 and t_3 become the optimal robust tours since they have edges with smaller ranges $u_e - l_e$ than t_1. For example, the difference between $l^T x$ for t_3 and t_1 is $(2\sqrt{2} + 2) - 4 \approx 0.83$. When $\Gamma = 1$, the maximum range among all edges of t_1 is

$\max\{(2.9-1), (2-1), 0, (2-1)\} = 1.9$ and that of t_3 is $\max\{(0, 1, 0, 1)\} = 1.$ The difference of the maximum range of edges (0.9) is greater than the difference of $l^T x$ (0.83). Thus, t_3 becomes the robust optimal solution when $\Gamma = 1.$

Computationally, (4) is a large integer program, and Atamtürk [3] studies how to strengthen formulations such as (4). While Atamtürk's methods are particularly favorable for mixed-integer formulations, i.e., those with binary and continuous variables, for pure binary formulations such as (4), Bertsimas and Sim show that the formulation decomposes into several "copies" of the deterministic integer program. This is attractive when efficient code for the deterministic problem is available.

In particular, Bertsimas and Sim show that (4) can be solved by solving $|E|+1$ deterministic TSPs. Assuming that the edge indices are ordered $1, 2, ..., |E|$ such that $u_1 - l_1 \geq u_2 - l_2 \geq \cdots \geq u_{|E|} - l_{|E|}$, and defining an artificial edge $|E|+1$ that has $u_{|E|+1} - l_{|E|+1} = 0$, equation (4) is equivalent to

$$\min_{e=1,\ldots,|E|+1} G^e, \tag{5}$$

where

$$G^e := \Gamma(u_e - l_e) + \min_{\mathbf{x}\in\mathcal{TSP}} \left\{ l^T x + \sum_{k=1}^{e} ((u_k - l_k) - (u_e - l_e)) x_k \right\}. \tag{6}$$

Calculating G^e for $e = 1, ..., |E|+1$ is a deterministic TSP for which high performance software is available, e.g., Concorde (Applegate et al. [1]). Finally, the optimal tour is recovered from the inner minimization of the minimum G^e.

Even though solving $|E|+1$ TSPs requires much less computational effort compared to solving (4) directly or enumerating all possible tours, practical problems may still be challenging due to the quadratic growth in the number of TSPs that must be solved (in the case of a complete or nearly complete network). In Section 5, we report computational times with this approach.

3.3. Robust deviation model

Montemanni et al. [32] apply a robust deviation criterion to iTSP specifically. Given a tour x and a realization of the uncertain data, the deviation of x with respect to the realization is defined to be the difference between the time of x and the optimal tour time for that realization. The maximum deviation of x is its maximum deviation over all realizations of the uncertain data. Montemanni et al. seek to find a tour x which has minimum maximum deviation. Based on previous results for the robust counterpart of other combinatorial optimization problems such as by Daniels and Kouvelis [18], Montemanni et al. show that the robust deviation of a given tour x is maximized under the realization in which all e in x have time u_e, while the remaining edges have time l_e. The authors refer to such a realization as a *realization induced by x*. This property allows iTSP with robust deviation criterion to be expressed in mathematical form as

$$\min_{x \in TSP} \left(u^T x - \min_{y \in TSP} \left\{ l^T y + \sum_{e \in E} (u_e - l_e) x_e y_e \right\} \right). \tag{7}$$

For the example shown in Figure 1, the robust deviations of each tour are as follows, where S_Tour_Ind(t) denotes the shortest tour under the realization induced by t:

Tour (t)	S_Tour_Ind(t)	Robust deviation of t
t_1	t_2	$4 - 2\sqrt{2} \approx 1.17$
t_2	t_3	1.9
t_3	t_2	2

Since the robust deviation of t_1 is smallest, it is the optimal robust tour, and the robust deviation is approximately 1.17.

The difficulty of this approach is that (7) is equivalent to a large integer program. The authors propose solution algorithms based on branch-and-bound, branch-and-cut, and Bender's decomposition. Among these three exact algorithms, Bender's decomposition performs best, but it still cannot always solve medium-sized problems $(n \approx 50)$ within reasonable computational time.

To partially overcome these computational challenges, the authors also propose heuristics. In Section 5, we use their heuristic, which they call HMU, for testing.

4. Modified-Soyster Model

We now present a simple variant of Soyster's model that allows one to control the model's conservatism and, when applied to iTSP, can be solved by optimizing a single deterministic TSP. The modification ensures that the model does not optimize to the extreme realizations of the uncertain data. To do this, we introduce a parameter $\gamma \in [0, 1]$ that controls the model's conservatism. When only the objective coefficients are uncertain and $\mathbf{x} \geq 0$, we have the modified-Soyster model

$$\begin{aligned} &\text{minimize } (\mathbf{c} + \gamma\hat{\mathbf{c}})^T \mathbf{x} \\ &\text{subject to } \mathbf{Ax} \leq \mathbf{b} \end{aligned}$$

for $\gamma \in [0, 1]$.

As with Soyster's original model, the modified-Soyster model can also be applied to integer programs and to iTSP in particular. However, one adjustment is required. While Soyster modeled the objective uncertainty as $c \pm \hat{c}$ with center c, our model $[l, u]$ does not possess a natural center. So we apply modified-Soyster to iTSP with $\gamma \in [0, 1]$ interpolating along the entire range between l and u:

$$\min_{x \in \mathcal{TSP}} l^T x + \gamma(u - l)^T x. \tag{8}$$

This is a deterministic TSP with travel times $(1-\gamma)l + \gamma u.$ In other words, γ interpolates between the most optimistic model of travel times, l, and the most pessimistic or conservative, u. In this sense, l and u are identified with c and $c + \hat{c}$ of modified-Soyster, respectively. A manager who wants to consider the worst-case travel times but be more optimistic overall would choose a low γ value, whereas a manager who wants to focus more on the worst-case u_e values while still considering the l_e values of the same edges would choose a high γ value.

The parameter γ plays a similar role to Γ in the model by Bertsimas and Sim described in Subsection 3.2. However, γ defines the range of all coefficients simultaneously – a balanced approach. On the other hand, Γ limits the number of coefficients allowed to change at one time – a focused approach. We will examine how taking a balanced versus a focused approach can impact the solutions in our computational experiments in Section 5. Modified-Soyster also allows the robust problem to remain relatively easy to solve, and offers some interesting theoretical properties, as described below in Subsection 4.1.

For our illustrative example with any $\gamma \in [0, 1],$ the robust time of each tour is as follows:

Tour	Robust time
t_1	$4 + \gamma(1.9 + 1 + 1 + 0)$
t_2	$2 + 2\sqrt{2} + \gamma(1.9 + 0 + 0 + 0)$
t_3	$2 + 2\sqrt{2} + \gamma(1 + 1 + 0 + 0)$

Thus, as γ varies, the robust optimal solutions are:

γ	Optimal robust tour
$[0, \sqrt{2} - 1]$	t_1
$[\sqrt{2} - 1, 1]$	t_2

4.1. Properties of modified-Soyster for iTSP

We now describe some interesting properties of modified-Soyster for iTSP. Let us first introduce some additional terminology. The *range of an edge e* is $u_e - l_e$, and the *range of a tour x* is the sum of all its edge ranges, i.e., $(u - l)^T x$. The *lower bound* and *upper bound* for x are $l^T x$ and $u^T x$, respectively. In particular, the upper bound for x is the sum of its lower bound and range. For a given $\gamma \in [0, 1]$, a *robust tour* is any optimal solution of (8).

The robust problem (8) may be interpreted as a multi-objective optimization problem that seeks to minimize the tour's lower bound and its γ-weighted range. As γ changes, one can expect a natural trade-off between the lower bounds and ranges of the resulting robust tours. We formalize this idea here.

Introducing two auxiliary scalar variables L and R (for "lower bound" and "range") and appealing to standard polyhedral theory, we write (8) as

$$\begin{aligned} \min\ & L + \gamma R \\ \text{s.t.}\ & x \in \text{convex.hull}(\mathcal{TSP}) \\ & L = l^T x \\ & R = (u - l)^T x. \end{aligned}$$

In the space of the variables (x, L, R), the feasible set is a polytope, and so its projection onto just (L, R) is also a polytope. In other words,

$$P := \left\{ (L, R) \in \Re^2 : \exists\ x \text{ satisfying } \begin{array}{l} x \in \text{convex.hull}(\mathcal{TSP}) \\ L = l^T x \\ R = (u - l)^T x \end{array} \right\}$$

is a polytope in $\Re^2$. Note that, in general, P has an exponential number of edges and vertices. Still, problem (8) can be written implicitly as the two-variable linear program

$$\min \; L + \gamma R \quad (LP_\gamma)$$

$$\text{s.t.} \; (L, R) \in P.$$

We let $\text{opt}(LP_\gamma)$ denote the set of optimal solutions of (LP_γ).

Consider the schematic of P shown in Figure 2. (LP_0) minimizes only the lower bound L, which implies $\text{opt}(LP_0) = \{(L_0, R_0)\}$. As γ increases, i.e., as more weight is put on minimizing the tour's range, $\text{opt}(LP_\gamma)$ is found on the bottom left boundary of P (shown in bold in the figure). In our context, γ is bounded above by 1, but one can imagine that, as $\gamma \to \infty$, (LP_γ) puts more and more weight on the range R. Eventually, after a certain threshold for γ, it holds that $\text{opt}(LP_\gamma) = \{(L_\infty, R_\infty)\}$.

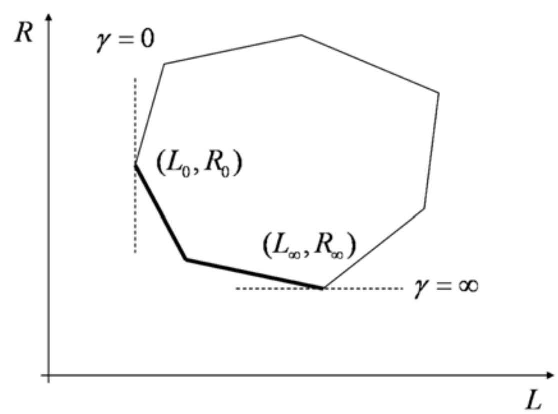

Figure 2. Schematic of polytope P.

Figure 2 provides intuition for Propositions 1-3 below. At a high level, Propositions 1-3 describe the following properties:

– As γ increases, the lower bound of a robust tour increases while its range decreases. This confirms a natural impact of increased conservatism: one must sacrifice the most optimistic outcome to guarantee a reduction in the pending uncertainty.

- The upper bound of a robust tour decreases as γ increases. Note that a tour's upper bound, which is the sum of its lower bound and range, is independent of γ and reflects the most pessimistic measure of a tour. In this sense, increased conservatism is guaranteed to mitigate the worst possible outcome.
- When γ is increased, the amount sacrificed in a robust tour's lower bound is always some fraction of the amount gained in the reduction of the tour's range. For a specific example, suppose $\gamma_1 = 0$ and $\gamma_2 = 0.5$. Then the amount sacrificed in the lower bound L by being more conservative is never more than 50% of what is gained by the reduction in the range R.

Each of the results deals with two choices of γ, namely $\gamma_1 < \gamma_2$. We use the notation (L_1, R_1) and (L_2, R_2) to indicate optimal pairs for $\text{opt}(LP_{\gamma_1})$ and $\text{opt}(LP_{\gamma_2})$, respectively. We first prove an important lemma.

Lemma 1. $\gamma_1(R_1 - R_2) \le L_2 - L_1 \le \gamma_2(R_1 - R_2)$.

Proof. Because $(L_1, R_1) \in \text{opt}(LP_{\gamma_1})$, it holds that

$$L_1 + \gamma_1 R_1 \le L_2 + \gamma_1 R_2 \Leftrightarrow \gamma_1(R_1 - R_2) \le L_2 - L_1.$$

Likewise, $(L_2, R_2) \in \text{opt}(LP_{\gamma_2})$ implies $L_2 - L_1 \le \gamma_2(R_1 - R_2)$.

Proposition 1. $L_1 \le L_2$ *and* $R_1 \ge R_2$.

Proof. Using the lemma, $\gamma_1\gamma_2^{-1}(R_1 - R_2) \le R_1 - R_2$, which guarantees $R_1 \ge R_2$. Then $L_2 - L_1 \ge 0$ since it is sandwiched between two nonnegative values.

Proposition 2. $L_2 + R_2 \le L_1 + R_1$.

Proof. Using the lemma and $\gamma_2 \le 1$, we have $L_2 - L_1 \le R_1 - R_2$, as desired.

Proposition 3. *Define* $\Delta L := L_2 - L_1 \geq 0$ *and* $\Delta R := R_1 - R_2 \geq 0$. *Then* $\gamma_1 \Delta R \leq \Delta L \leq \gamma_2 \Delta R$.

Proof. This is just a restatement of the lemma with Proposition 1.

Figure 3 illustrates Propositions 1 and 2 for a specific instance of iTSP. The instance is based on `swiss42.tsp` from TSPLIB. For various values of $\gamma \in [0, 1]$ the optimal values L and R of (LP_γ) are plotted; the sum $L + R$ is also plotted. Proposition 1 is illustrated by the fact that L increases with γ, while R decreases as γ increases. The fact that $L + R$ decreases with increasing γ demonstrates Proposition 2.

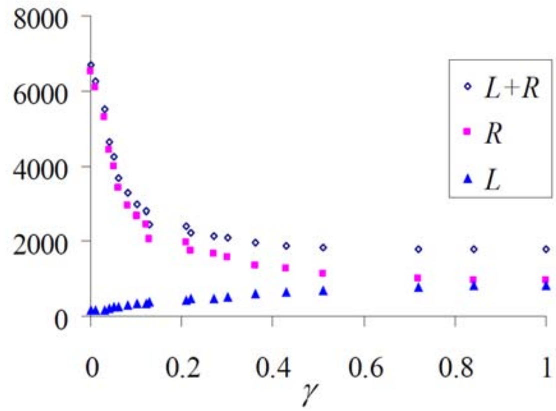

Figure 3. Example of the solutions provided by modified-Soyster for iTSP.

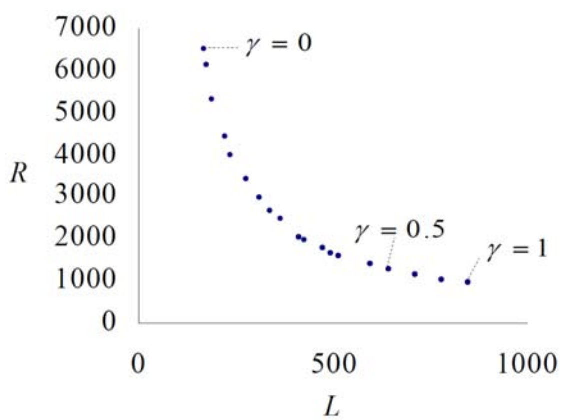

Figure 4. Example of the solutions provided by modified-Soyster for iTSP.

In addition, Figure 4 depicts a scatter plot of the optimal pairs (L, R) for the same example, and the γ values corresponding to several points are highlighted $(\gamma = 0.0, 0.5, 1.0)$. One can view the points as the vertices of the polyhedron P. In addition, Proposition 3 has the following interpretation in terms of the figure: the slope between any two pairs is less than or equal to –1. (However, we caution the reader that the aspect ratio of Figure 4 is not 1 : 1, and so some slopes may appear greater than –1.)

Propositions 1-3 do not appear to have direct analogs for the Bertsimas-Sim model. As an example, consider Figure 5. For missing edges, we assume sufficiently large travel times so as to make them part of no optimal solution. Then there are only two possible tours, $t_1 = (1, 2, 3, 4, 5, 1)$ and $t_2 = (1, 2, 4, 3, 5, 1)$. Table 1 shows the solutions and associated values for Bertsimas-Sim. It is clear none of the analogs of Propositions 1-3 are satisfied in this case. The main reason is Bertsimas-Sim's focused approach of considering only the ranges of a given number Γ of edges, while the ranges of remaining edges are in essence ignored. In contrast, the balanced approach of modified-Soyster seems to be critical in establishing results such as Propositions 1-3.

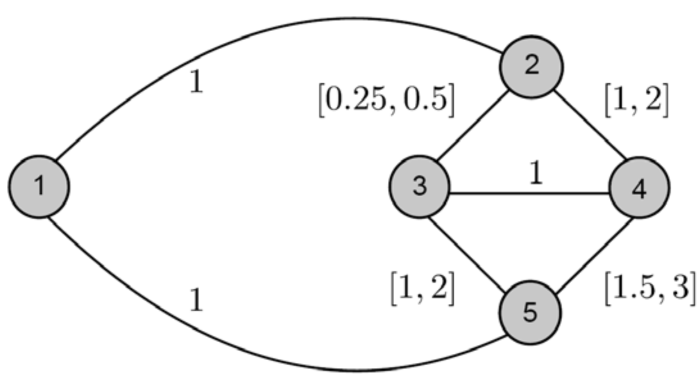

Figure 5. Second example of robust TSP problem with interval times.

Table 1. Optimal tours and associated times with the Bertsimas-Sim model

Γ	Optimal robust tour x	Lower bound $l^T x$	Range $(u-l)^T x$	Upper bound $u^T x$
0	t_1	4.75	1.75	6.5
1	t_2	5	2	7
≥ 2	t_1	4.75	1.75	6.5

Finally, we note that Propositions 1-3 are about the behaviors of the solution and associated times as the degree of conservatism changes. Since the level of conservatism is not controllable in the model by Montemanni et al., there are no analogs to these propositions for the robust deviation model.

5. Computational Experiments

In this section, we compare the four models computationally. We first discuss our test instances and computing environment. Then we examine computational times, which demonstrate that modified-Soyster enjoys relatively quick solution times compared to the other models, while still incorporating uncertainty and allowing one to control the level of conservatism. Next, we show that the solution provided by modified-Soyster does reasonably well at creating solutions of high quality for other objectives if γ is chosen appropriately. In this regard, one may also consider modified-Soyster a quick heuristic for *more time-consuming* uncertainty objectives. Lastly, we demonstrate a nice property of the solutions found by modified-Soyster as compared with Bertsimas-Sim.

5.1. Test instances and computational environment

We test 72 instances of iTSP created by Montemanni et al. [32] based on 12 problems in TSPLIB. We thank the authors for sharing their instances. Given a TSPLIB problem with travel times c_e and a parameter $\beta \in (0, 1)$, they form an instance of iTSP by randomly choosing integer $l_e \in [(1-\beta)c_e, c_e]$ and integer $u_e \in [c_e, (1+\beta)c_e]$. In particular, the l_e and u_e

values may vary from c_e by different amounts, and the β parameter determines the maximum distance these values may vary from c_e. The 72 instances used here consist of 3 random instances with $\beta = 0.25$ and 3 with $\beta = 0.5$ for each of the 12 TSPLIB instances.

We employ Concorde (Applegate et al. [1]) to solve the deterministic TSPs. For each model, the code was written in MATLAB with calls to Concorde on an Intel Pentium D 3.20 GHz machine.

In the following two subsections, we will actually consider two specifications of modified-Soyster as well as two specifications of Bertsimas-Sim. The full list of models considered is:

1. Soyster's "pure pessimistic" model with all travel times set to u_e (M_1);
2. "optimistic" modified-Soyster with $\gamma = 5/n$ (M_2);
3. "pessimistic" modified-Soyster with $\gamma = 0.5$ (M_3);
4. the robust-deviation model of Montemanni et al. (M_4);
5. "optimistic" Bertsimas-Sim with $\Gamma = 5$ (M_5);
6. "pessimistic" Bertsimas-Sim with $\Gamma = n/2$ (M_6).

Note that modified-Soyster for a given γ is not directly comparable to Bertsimas-Sim with a certain Γ. However, we choose $\gamma = 5/n$ and $\Gamma = 5$ to reflect low levels of conservatism (i.e., more optimism) and $\gamma = 0.5$ and $\Gamma = n/2$ to represent high level of conservatism (i.e., more pessimism).

5.2. Computational times

Table 2 shows the average run time (in seconds) by problem size n for the 72 instances. As mentioned in Subsection 3.3, we will use Montemanni et al.'s HMU heuristic for solving their robust deviation model. This table makes clear that modified-Soyster is competitive in terms of speed with

Soyster (as one would expect) and slightly faster than robust deviation. The more notable observation is that Bertsimas-Sim is quite time-consuming especially for the larger problem sizes.

5.3. Quality of modified-Soyster for other models

We investigate the quality of the tours provided by modified-Soyster in terms of the more time-consuming objectives. For each of the 72 test instances, we solved the instance with both modified-Soyster specifications $(M_2$ and $M_3)$ and three other models $(M_4$ through $M_6)$. We saved the tours from M_2 and M_3 and the optimal values from the remaining. We use $v_j(M_i)$ to represent the objective value of a tour found with model i when evaluated using the objective function for M_j. Then, for a model pair (i, j) with $i \in \{2, 3\}$ and $j \in \{4, 5, 6\}$, we calculate the modified-Soyster optimality gap for that instance:

$$\left(\frac{v_j(M_i)}{v_j(M_j)} - 1\right) \times 100\%.$$

Table 2. Average computational times (in seconds) by problem size

n	# instances	Soyster	Mod-Soyster $(\gamma = 5/n)$	Mod-Soyster $(\gamma = 0.5)$	Robust deviation	Bert-Sim $(\Gamma = 5)$	Bert-Sim $(\Gamma = n/2)$
17	6	0.0	0.0	0.0	0.2	6.2	6.1
21	6	0.0	0.0	0.0	0.1	4.0	4.1
24	6	0.0	0.0	0.0	0.2	9.2	9.2
26	6	0.0	0.0	0.0	0.2	14.9	14.7
42	12	0.1	0.1	0.1	0.4	61.7	61.5
48	12	0.1	0.1	0.1	0.5	144.8	141.3
58	6	0.2	0.3	0.2	1.2	461.0	463.2
120	6	0.6	0.5	2.1	6.0	4,738.5	4,793.9
175	6	2.4	2.4	0.5	7.2	33,274.3	33,338.0
180	6	1.2	0.6	1.2	6.0	19,702.1	19,841.4

Table 3 shows these optimality gaps averaged over the 36 instances with $\beta = 0.25$, and Table 4 shows the same except for $\beta = 0.50$.

Table 3. Average optimality gaps for modified-Soyster with respect to other models for $\beta = 0.25$

	Modified-Soyster ($\gamma = 5/n$)	Modified-Soyster ($\gamma = 0.5$)
Robust deviation	7.64%	1.19%
Bertsimas-Sim ($\Gamma = 5$)	0.44%	0.45%
Bertsimas-Sim ($\Gamma = n/2$)	1.31%	0.35%

Table 4. Average optimality gaps for modified-Soyster with respect to other models for $\beta = 0.50$

	Modified-Soyster ($\gamma = 5/n$)	Modified-Soyster ($\gamma = 0.5$)
Robust deviation	16.24%	1.46%
Bertsimas-Sim ($\Gamma = 5$)	2.13%	1.90%
Bertsimas-Sim ($\Gamma = n/2$)	5.70%	0.98%

Pessimistic modified-Soyster performs within 1% of pessimistic Bertsimas-Sim for both β values, and optimistic modified-Soyster performs within 2.13% of optimistic Bertsimas-Sim across both tables. This indicates that the choice of γ and Γ provide comparable levels of conservatism. It is surprising that pessimistic modified-Soyster performs closer to optimistic Bertsimas-Sim than optimistic modified-Soyster in Table 4, but it is by a fairly small margin.

The largest gaps in both tables are between optimistic modified-Soyster and the robust deviation models, indicating robust deviation is a rather pessimistic approach. We observe that pessimistic modified-Soyster performs within 1.46% of robust deviation across both β values, indicating that modified-Soyster with a high γ can be potentially be used as an alternative for robust deviation. Even though the runtimes are not drastically different, this may be preferred because of its simpler implementation.

Overall, Tables 3 and 4 illustrate that the tours provided by modified-Soyster often do well as solutions of the other models. We propose that modified-Soyster, with an appropriate choice of γ, could be used as a quick heuristic for other uncertainty models, particularly Berstimas-Sim.

5.4. Further comparison between modified-Soyster and Bertsimas-Sim

To close this section, we provide one more computational comparison between modified-Soyster and Bertsimas-Sim. In particular, we would like to contrast the focused conservatism of Bertsimas-Sim (i.e., Γ edges take on their u_e values, while the remaining stay at l_e) and the balanced conservatism of modified-Soyster (i.e., each edge takes on the value $l_e + \gamma(u_e - l_e)$). We claim that, at least from the point of view of worst-case performance of a tour, the balanced conservatism of modified-Soyster may be more appropriate for instances of iTSP having many edges with similar l_e values but larger and more varied u_e values.

To support this claim, we create and solve 36 such instances - one derived from each of the 36 instances of iTSP with $\beta = 0.5$ used in the preceding subsections. Although we admit the new instances are somewhat ad hoc, they more readily illustrate the case of similar l_e values and more extreme u_e values.

Our construction is as follows. Let (l, u) be the data of a given instance, and define $s := \min_{e \in E} l_e$ to be the smallest optimistic travel time. For each $e \in E$, also choose a random integer $\tilde{l}_e \in [s, \min\{2s, u_e\}]$, and define $\tilde{u}_e = u_e$. The data for the new iTSP is then $(\tilde{l}, \tilde{u})$. This new instance is highly likely to have small variation in $\tilde{l}$ and larger, more varied values in $\tilde{u}$.

Each of the new 36 problems is solved with optimistic modified-Soyster $(\gamma = 5/n)$ and optimistic Bertsimas-Sim $(\Gamma = 5)$, and the two optimal tours are saved. Then the optimality gaps for these tours are computed with respect to Soyster's model (whose optimal value is separately calculated). Table 5 presents the average optimality gaps.

Table 5. Average optimality gaps for modified-Soyster and Bertsimas-Sim with respect to Soyster's model for new set of 36 instances

Modified-Soyster $(\gamma = 5/n)$	Bertsimas-Sim $(\Gamma = 5)$
11.3%	33.9%

Table 5 shows the benefit of balanced conservatism on such instances. By incorporating all u_e values into its calculations, modified-Soyster produces a tour that protects much better against the worst case than Bertsimas-Sim. In contrast, by focusing on just Γ worst-case u_e values per tour, Bertsimas-Sim produces a tour that may do poorly in the worst case. Since modified-Soyster takes the full range of $u_e - l_e$ into account for all edges, it clearly changes the choice of edges for comparable levels of optimism.

6. Managerial Insights and Future Work

From a managerial perspective, modified-Soyster is a good model to use for instances of iTSP. It allows managers to designate the level of optimism or conservatism in the solutions and does so without increasing computational difficulty relative to the TSP. This can be quite useful in practice. For example, if a manager is creating a route that will be driven during off-peak travel times, this may make him or her optimistic in terms of the travel times that will occur. Among edges with similar best case, or l_e, travel times, the manager would prefer those with lower u_e travel times in case some traffic does occur. Thus, such a manager may use modified-Soyster with a low value of γ and achieve similar runtimes as with the classical TSP. Our experiments have shown that Bertsimas-Sim, the other approach that allows for control of conservatism, does so in a way that is much more time consuming computationally and offers solutions that can differ greatly in terms of worst-case evaluation. Our experiments also suggest that managers may use modified-Soyster even when interested in other models. Modified-Soyster, with the appropriate choice of γ, can yield solutions that perform well in terms of other objectives, while being easy to implement as well as quick to solve. A manager may also like some of the structural properties for modified-Soyster listed in Subsection 4.1, which highlight predictable effects of changing γ. For example, by increasing γ, the manager can be sure that he or she will obtain a tour with better worst-case performance. Also, a higher γ guarantees a tour having a narrower range of potential time outcomes, thus reducing the manager's uncertainty.

In the future, we would like to extend our results to more complicated but related problems. Specifically, we are interested in modifying and extending our results for the asymmetric TSP, where the travel time on an edge can differ depending on the direction being traveled. We are also interested in variants of the iTSP where the value of γ can be an edge specific value γ_e.

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