Chapter 8
Copositive Programming

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8.1 Introduction

A symmetric matrix S is *copositive* if $y^T S y \geq 0$ for all $y \geq 0$, and the set of all copositive matrices, denoted $\mathcal{C}^*$, is a closed, pointed, convex cone; see [25] for a recent survey. Researchers have realized how to model many NP-hard optimization problems as *copositive programs*, that is, programs over $\mathcal{C}^*$ for which the objective and all other constraints are linear [7, 9, 13, 16, 32–34]. This makes copositive programming NP-hard itself, but the models are nevertheless interesting because copositive programs are convex, unlike the problems which they model. In addition, $\mathcal{C}^*$ can be approximated up to any accuracy using a sequence of polyhedral-semidefinite cones of ever larger sizes [13, 30], so that an underlying NP-hard problem can be approximated up to any accuracy if one is willing to spend the computational effort.

In actuality, most of these NP-hard problems are modeled as linear programs over the dual cone $\mathcal{C}$ of *completely positive* matrices, that is, matrices Y that can be written as the sum of rank-1 matrices yy^T for $y \geq 0$ [4]. These programs are called *completely positive programs*, and the aforementioned copositive programs are constructed using standard duality theory.

Currently the broadest class of problems known to be representable as completely positive programs are those with nonconvex quadratic objective and linear constraints over binary and continuous variables [9]. In addition, complementarity constraints on bounded, nonnegative variables can be incorporated. In this chapter, we recount and extend this result using the more general notion of matrices that are *completely positive over a closed, convex cone*.

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Recall the concept of a *linear conic program over* $\mathcal{K}$, where $\mathcal{K}$ is assumed to be a closed, convex cone [2]:

$$\min\{c^T x : Ax = b, x \in \mathcal{K}\}. \tag{8.1}$$

The standard form (8.1) is quite general. For example, it can be used to model mixed linear, second-order-cone, semidefinite programs with free variables. In such cases, $\mathcal{K}$ is the Cartesian product of a nonnegative orthant, second-order cones, positive semidefinite cones, and a Euclidean space. The form of (8.1) is also critical for the design, analysis, and implementation of algorithms such as the simplex method when $\mathcal{K}$ is a nonnegative orthant and interior-point methods more generally. For example, the iteration complexity of an interior-point method for (8.1) will depend on the self-concordancy barrier parameter for $\mathcal{K}$ [29]. Loosely speaking, understanding $\mathcal{K}$ is key for optimizing (8.1).

When faced with a new optimization problem, it thus seems prudent to determine if that problem can be modeled as a linear conic program over some (hopefully) well understood $\mathcal{K}$.

In this chapter, we show that any NP-hard *nonconvex quadratic conic program*

$$\nu_* := \min\left\{x^T Qx + 2\,c^T x : Ax = b, x \in \mathcal{K}\right\} \tag{8.2}$$

can be modeled as an explicit linear conic program over the closed, convex cone $\mathcal{C}$ of matrices that are completely positive over $\Re_+ \times \mathcal{K}$, i.e., matrices Y that can be written as the sum of rank-1 matrices yy^T with $y \in \Re_+ \times \mathcal{K}$.[1] We also extend this result to include certain types of nonconvex quadratic constraints. While $\mathcal{C}$ may be harder to understand than $\mathcal{K}$ [3, 27], our approach provides a convex formulation for (8.2). For example, strong relaxations of $\mathcal{C}$ can be used to compute high quality lower bounds on ν_*. In addition, (8.2) motivates the study of such cones $\mathcal{C}$, particularly for common $\mathcal{K}$ such as the Cartesian product of a nonnegative orthant, second-order cones, semidefinite cones, and a Euclidean space.[2]

The equivalence of (8.2) with a linear conic program over $\mathcal{C}$ is actually based on the characterization of a certain convex hull in $\mathcal{C}$. This is the core of Sect. 8.2. We also extend in Sect. 8.2 the main convex hull result in several directions, e.g., the result is extended to incorporate certain types of nonconvex quadratic constraints in the variable x. Section 8.3 applies the convex hull results to (8.2) and related optimization problems and also presents some basic duality results. Section 8.4

[1] The paper [20] has established a similar result concurrently with this chapter, and the paper [19] studies the generalized notion of copositivity over an arbitrary set, analyzing important properties of the resulting convex cone.

[2] If $\mathcal{K}$ is a semidefinite cone, than $\mathcal{K}$ must be encoded in its "vectorized" form to match the development in this chapter. For example, the columns of a $d \times d$ semidefinite matrix could be stacked into a single, long column vector of size d^2, and then the CP matrices yy^T over $\Re_+ \times \mathcal{K}$ would have size $(d^2+1)\times(d^2+1)$.

discusses techniques for working with $\mathcal{C}$, particularly when $\mathcal{K}$ is the nonnegative orthant $\Re^n_+$, which has been the most studied case. Section 8.5 concludes the chapter with a few applications.

We make two important remarks. First, this chapter is not meant to be a complete survey of copositive programming. For this, please see the excellent, recent paper [17]. Instead, we intend only to give an introduction to some of the main results in the area, tainted with our own biases. Second, most research has examined the case when $\mathcal{K} = \Re^n_+$. Our hope is that this chapter will stimulate further investigations for more general $\mathcal{K}$, especially when $\mathcal{K}$ is the Cartesian product of a nonnegative orthant, second-order cones, semidefinite cones, and a Euclidean space.

8.2 Convex Hull Results

We first prove the main convex hull results that will be the basis of Sect. 8.3.

8.2.1 Main Result

Define the feasible set and recession cone of (8.2) to be

$$\mathcal{L} := \{x \in \mathcal{K} : Ax = b\}, \qquad \mathcal{L}_\infty := \{d \in \mathcal{K} : Ad = 0\},$$

where $A \in \Re^{m\times n}$ and $b \in \Re^m$. We assume $\mathcal{L} \neq \emptyset$, i.e., (8.2) is feasible. The set of *completely positive matrices* over the cone $\Re_+ \times \mathcal{K}$ is defined as

$$\mathcal{C} := \left\{ \sum_k \binom{\zeta^k}{z^k}\binom{\zeta^k}{z^k}^T : \binom{\zeta^k}{z^k} \in \Re_+ \times \mathcal{K} \right\}.$$

$\mathcal{C}$ is closed because $\Re_+ \times \mathcal{K}$ is closed [35, Lemma 1]. The representation of $Y \in \mathcal{C}$ in terms of (ζ^k, z_k) is called a *completely positive decomposition*, and the decomposition is *proper* if $(\zeta^k, z_k) \neq 0$ for all k. Also define the following two subsets of $\mathcal{C}$:

$$\mathcal{L}^1 := \left\{ \binom{1}{x}\binom{1}{x}^T : x \in \mathcal{L} \right\}, \qquad \mathcal{L}^0_\infty := \left\{ \binom{0}{d}\binom{0}{d}^T : d \in \mathcal{L}_\infty \right\}.$$

One can think of $\mathcal{L}^1$ and $\mathcal{L}^0_\infty$ as quadratic versions of $\mathcal{L}$ and $\mathcal{L}_\infty$, respectively. The following proposition relates these two sets, where conv$(\cdot)$ denotes the convex hull, cone$(\cdot)$ denotes the convex conic hull, and clconv$(\cdot)$ denotes the closure of the convex hull.

Proposition 8.1. $\operatorname{conv}(\mathcal{L}^1) + \operatorname{cone}(\mathcal{L}^0_\infty) \subseteq \operatorname{clconv}(\mathcal{L}^1)$.

Proof. We represent an arbitrary element $Y \in \text{conv}(\mathcal{L}^1) + \text{cone}(\mathcal{L}^0_\infty)$ via a finite sum

$$Y = \sum_{k \in P} \lambda_k \binom{1}{x^k}\binom{1}{x^k}^T + \sum_{k \in Z} \binom{0}{d^k}\binom{0}{d^k}^T$$

where P and Z are finite index sets, $x^k \in \mathcal{L}$, $d^k \in \mathcal{L}_\infty$, $\lambda_k > 0$, and $\sum_{k \in P} \lambda_k = 1$. Next let $\bar{x} \in \mathcal{L}$ be fixed, and let $\epsilon_k > 0$ for $k \in P \cup Z$ be fixed such that $\sum_{k \in P} \epsilon_k = \sum_{k \in Z} \epsilon_k$. Define

$$Y^\epsilon := \sum_{k \in P} (\lambda_k - \epsilon_k) \binom{1}{x^k}\binom{1}{x^k}^T + \sum_{k \in Z} \epsilon_k \binom{1}{\bar{x} + d^k/\sqrt{\epsilon_k}}\binom{1}{\bar{x} + d^k/\sqrt{\epsilon_k}}^T.$$

If ϵ is sufficiently close to the zero vector, then $Y^\epsilon \in \text{conv}(\mathcal{L}^1)$. Taking a sequence of such ϵ converging to zero, we see $Y^\epsilon \to Y$, which implies $Y \in \text{clconv}(\mathcal{L}^1)$. □

We next introduce a third subset of C:

$$\mathcal{R} := \left\{ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in C : \begin{array}{c} Ax = b \\ \text{diag}(AXA^T) = b \circ b \end{array} \right\}.$$

Here, the symbol $\circ$ indicates the Hadamard product of vectors. Note that $\mathcal{R}$ is closed and convex. By standard relaxation techniques, $\mathcal{L}^1$ is contained in $\mathcal{R}$. So $\text{clconv}(\mathcal{L}^1) \subseteq \mathcal{R}$. We can prove more.

Proposition 8.2. $\text{clconv}(\mathcal{L}^1) \subseteq \mathcal{R} \subseteq \text{conv}(\mathcal{L}^1) + \text{cone}(\mathcal{L}^0_\infty)$.

Proof. The first containment holds by construction. To show the second, let

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \sum_k \binom{\zeta^k}{z^k}\binom{\zeta^k}{z^k}^T \tag{8.3}$$

be an arbitrary element of $\mathcal{R}$ with proper completely positive decomposition. Partition the summands of (8.3) via the index sets $P := \{k : \zeta^k > 0\}$ and $Z := \{k : \zeta^k = 0\}$. We claim: (i) $k \in P$ implies $z^k/\zeta^k \in \mathcal{L}$; (ii) $k \in Z$ implies $z^k \in \mathcal{L}_\infty$.

To prove both parts of the claim, we need a technical result. From (8.3), we see

$$\sum_k (\zeta^k)^2 = 1. \tag{8.4}$$

Moreover, since $Ax = b$ and $\text{diag}(AXA^T) = b \circ b$, we have

$$\begin{aligned} b &= \sum_k \zeta^k (Az^k) \\ b \circ b &= \sum_k \text{diag}(Az^k(z^k)^T A^T) = \sum_k (Az^k) \circ (Az^k). \end{aligned} \tag{8.5}$$

Thus

$$\left(\sum_k \zeta^k (Az^k)\right) \circ \left(\sum_k \zeta^k (Az^k)\right) = \left(\sum_k (\zeta^k)^2\right)\left(\sum_k \left(Az^k\right) \circ \left(Az^k\right)\right)$$

and so by the equality-case of the Cauchy–Schwarz inequality, there exists $\delta \in \Re^m$ such that, for all k,

$$\zeta^k \delta = Az^k. \tag{8.6}$$

Claimed item (ii) follows directly from (8.6) and the fact that $\zeta^k = 0$ when $k \in Z$. To prove (i), it suffices to show $\delta = b$. Indeed, (8.4), (8.5), and (8.6) imply

$$b = \sum_k \zeta^k (Az^k) = \sum_k \zeta^k (\zeta^k \delta) = \delta.$$

With claims (i) and (ii) established, we now complete the proof of the theorem. Taking $\lambda_k := (\zeta^k)^2$, $x^k := z^k/\zeta^k$ for all $k \in P$, and $d^k := z^k$ for all $k \in Z$, we can write the completely positive decomposition (8.3) in the more convenient form

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \sum_{k \in P} \lambda_k \binom{1}{x^k}\binom{1}{x^k}^T + \sum_{k \in Z} \binom{0}{d^k}\binom{0}{d^k}^T \tag{8.7}$$

where $\lambda_k > 0$, $\sum_{k \in P} \lambda_k = 1$, $x^k \in \mathcal{L}$, and $d^k \in \mathcal{L}_\infty$. □

Propositions 8.1 and 8.2 combine to give the following key theorem.

Theorem 8.1. $\mathcal{R} = \mathrm{clconv}(\mathcal{L}^1)$.

The proofs for Proposition 8.1–8.2 and Theorem 8.1 have been inspired by [5, 9].

8.2.2 Additional Implied Constraints

Because $\mathcal{R}$ is contained in the positive semidefinite cone, the constraints $Ax = b$ and $\mathrm{diag}(AXA^T) = b \circ b$ actually imply more. The following proposition and corollary were proved in [10], and very closely related results appear in [21].

Proposition 8.3. *Suppose*

$$Y = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0$$

and define $M := \left(b, -A\right)$ to be the matrix formed by concatenating b and $-A$. Then the following are equivalent:

(i) $Ax = b$, $\mathrm{diag}(AXA^T) = b \circ b$.
(ii) $MYM^T = 0$.
(iii) $MY = 0$.

Proof. We will use the following equations:

$$MY = \begin{pmatrix} b, & -A \end{pmatrix} \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \begin{pmatrix} b - Ax & bx^T - AX \end{pmatrix}$$

$$MYM^T = MY \begin{pmatrix} b^T \\ -A^T \end{pmatrix} = bb^T - Axb^T - bx^T A^T + AXA^T. \tag{8.8}$$

$(i) \Rightarrow (ii)$: We have $MYM^T = AXA^T - bb^T$ with zero diagonal. Since Y is positive semidefinite, so is MYM^T. Hence, $MYM^T = 0$.

$(ii) \Rightarrow (iii)$: Let $Y = VV^T$ be a Gram representation of Y, which exists because $Y \succeq 0$. We have $0 = \text{trace}(MYM^T) = \text{trace}(MVV^T M^T) = \|MV\|_F^2$, where F indicates the Frobenius norm, and so $MV = 0$, which implies $MY = (MV)V^T = 0$.

$(iii) \Rightarrow (i)$: $Ax = b$ is clear from (8.8). Also $AX = bx^T$, which implies $AXA^T = bb^T$, so $\text{diag}(AXA^T) = b \circ b$. □

Because $\mathcal{C}$ is a subset of the positive semidefinite matrices, Proposition 8.3 implies that the constraints $MYM^T = 0$ and $MY = 0$ are redundant for $\mathcal{R}$.

Corollary 8.1. *Define $M := \begin{pmatrix} b, & -A \end{pmatrix}$. Then every $Y \in \mathcal{R}$ satisfies the additional equations $MYM^T = 0$ and $MY = 0$.*

Proposition 8.3 also establishes that $\mathcal{R}$ lacks interior, where by definition $Y \in \mathcal{R}$ is interior if $Y \in \text{int}(\mathcal{C})$. Since $\mathcal{C}$ is contained in the positive semidefinite matrices, $\text{int}(\mathcal{C})$ is contained in the positive definite matrices. As every $Y \in \mathcal{R}$ has nontrivial null space as demonstrated by $MY = 0$, every $Y \in \mathcal{R}$ is not positive definite, i.e., $\mathcal{R} \cap \text{int}(\mathcal{C}) = \emptyset$.

8.2.3 Extraneous Variables

It is sometimes possible to eliminate the variable x in $\mathcal{R}$ and consequently express $\mathcal{R}$ in terms of a slightly smaller cone $\mathcal{C}_0$ instead of $\mathcal{C}$. The cone $\mathcal{C}_0$ is the set of matrices that are completely positive over $\mathcal{K}$:

$$\mathcal{C}_0 := \left\{ \sum_k z^k (z^k)^T : z^k \in \mathcal{K} \right\}.$$

The key property we require is the following:

$$\exists \; y \in \Re^m \text{ s.t. } A^T y \in \mathcal{K}^*, \; b^T y = 1, \tag{8.9}$$

where $\mathcal{K}^* := \{s \in \Re^n : s^T x \geq 0 \; \forall \; x \in \mathcal{K}\}$ is the dual cone of $\mathcal{K}$. In this subsection, we assume (8.9) and define

$$\alpha := A^T y \in \mathcal{K}^*. \tag{8.10}$$

A direct consequence of (8.9) is that $\alpha^T x = 1$ is redundant for $\mathcal{L}$. So Theorem 8.1 establishes that

$$\mathcal{R} = \text{clconv}(\mathcal{L}^1) = \text{clconv}\left(\mathcal{L}^1 \cap \left\{x : \alpha^T x = 1\right\}\right)$$
$$= \mathcal{R} \cap \left\{ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} : \begin{array}{c} \alpha^T x = 1 \\ \alpha^T X \alpha = 1 \end{array} \right\}.$$

In other words, $\alpha^T x = 1$ and $\alpha^T X\alpha = 1$ are redundant for $\mathcal{R}$. Now applying Proposition 8.3, we see

$$\left(1, -\alpha^T\right)\begin{pmatrix} 1 \\ x \end{pmatrix} = 0 \quad \Longleftrightarrow \quad x = X\alpha$$

and so

$$\mathcal{R} = \left\{ \begin{pmatrix} 1 & \alpha^T X \\ X\alpha & X \end{pmatrix} \in \mathcal{C} : \begin{array}{c} AX\alpha = b \\ \text{diag}(AXA^T) = b \circ b \\ \alpha^T X\alpha = 1 \end{array} \right\}$$

Finally, the equations $\alpha^T X\alpha = 1$ and

$$\begin{pmatrix} 1 & \alpha^T X \\ X\alpha & X \end{pmatrix} = \left(\alpha \; I\right)^T X \left(\alpha \; I\right)$$

along with $\alpha \in \mathcal{K}^*$ demonstrate that

$$X \in \mathcal{C}_0 \quad \Longrightarrow \quad \begin{pmatrix} 1 & \alpha^T X \\ X\alpha & X \end{pmatrix} \in \mathcal{C}.$$

Moreover, the converse holds because the bottom-right $n \times n$ principal submatrix of a matrix in $\mathcal{C}$ is necessarily in $\mathcal{C}_0$. Hence:

Theorem 8.2. *Suppose (8.9) holds, and define α via (8.10). Then*

$$\mathcal{R} = \left\{ \begin{pmatrix} 1 & \alpha^T X \\ X\alpha & X \end{pmatrix} : X \in \mathcal{R}_0 \right\}$$

where

$$\mathcal{R}_0 := \left\{ X \in \mathcal{C}_0 : \begin{array}{c} AX\alpha = b \\ \text{diag}(AXA^T) = b \circ b \\ \alpha^T X\alpha = 1 \end{array} \right\}.$$

In addition, $\mathcal{R}_0 := \text{clconv}\left(\{xx^T : x \in \mathcal{L}\}\right)$.

Besides a more compact representation, an additional benefit of $\mathcal{R}_0$ over $\mathcal{R}$ is that $\mathcal{R}_0$ may have an interior, whereas $\mathcal{R}$ never does (see discussion in previous subsection). This is an important feature of $\mathcal{R}_0$ since the existence of an interior is generally a benefit both theoretically and computationally. However, it seems difficult to establish general conditions under which $\mathcal{R}_0$ is guaranteed to have an interior. This is due in part to the general data (A, b) but also to the fact that not much is known about the structure of the interior of C_0. The paper [18] studies the case when $\mathcal{K} = \Re^n_+$.

8.2.4 Quadratic Constraints

Consider a quadratic function $x^T F x + 2 f^T x$, and define

$$\phi_* := \min_{x \in \mathcal{L}} \left(x^T F x + 2 f^T x \right), \qquad \phi^* := \max_{x \in \mathcal{L}} \left(x^T F x + 2 f^T x \right).$$

We wish to establish conditions under which a result similar to Theorem 8.1 holds for the further constrained feasible set

$$\mathcal{L}' := \mathcal{L} \cap \{x : x^T F x + 2 f^T x = \phi_*\}.$$

In addition to the sets $\mathcal{L}_\infty$ and $\mathcal{L}^0_\infty$ already defined in Sect. 8.2.1, define

$$(\mathcal{L}')^1 := \left\{ \binom{1}{x}\binom{1}{x}^T : x \in \mathcal{L}' \right\}$$

$$\mathcal{R}' := \mathcal{R} \cap \left\{ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} : F \bullet X + 2 f^T x = \phi_* \right\},$$

where $\bullet$ indicates the trace inner product.

Theorem 8.3. *Suppose both ϕ_* and ϕ^* are finite and there exists $\bar{x} \in \mathcal{L}'$ such that $d^T(F\bar{x} + f) = 0$ for all $d \in \mathcal{L}_\infty$. Then $\mathcal{R}' = \operatorname{clconv}\left((\mathcal{L}')^1\right)$.*

Proof. Analogous to Propositions 8.1 and 8.2, we argue

$$\operatorname{conv}((\mathcal{L}')^1) + \operatorname{cone}(\mathcal{L}^0_\infty) \subseteq \operatorname{clconv}((\mathcal{L}')^1)$$

$$\operatorname{clconv}((\mathcal{L}')^1) \subseteq \mathcal{R}' \subseteq \operatorname{conv}((\mathcal{L}')^1) + \operatorname{cone}(\mathcal{L}^0_\infty).$$

To prove the first, we imitate the proof of Proposition 8.1, except here we specifically choose $\bar{x}$ as hypothesized. The only thing to check is that $Y^\epsilon \in \operatorname{conv}((\mathcal{L}')^1)$ or, more specifically, that $\bar{x} + d^k/\sqrt{\epsilon_k} \in \mathcal{L}'$. It suffices to show $\bar{x} + d \in \mathcal{L}'$ for all $d \in \mathcal{L}_\infty$. We already know $\bar{x} + d \in \mathcal{L}$. It remains to show

$$(\bar{x} + d)^T F(\bar{x} + d) + 2 f^T(\bar{x} + d) = \phi_* \iff$$

$$d^T F d + 2 d^T (F\bar{x} + f) = 0 \iff$$

$$d^T F d = 0$$

which is true since $-\infty < \phi_*$ and $\phi^* < \infty$; otherwise, d would be a direction of recession to drive $x^T F x + 2 f^T x$ to $-\infty$ or ∞.

To prove the second, we imitate the proof of Proposition 8.2. The inclusion $\mathrm{conv}((\mathcal{L}')^1) \subseteq \mathcal{R}'$ is clear. For the second inclusion, the representation (8.7) holds without change. We next show that the constraint $F \bullet X + 2 f^T x = \phi_*$ implies $(x^k)^T F x^k + 2 f^T x^k = \phi_*$ for all $k \in P$. From the previous paragraph, we know $(d^k)^T F d^k = 0$ for all $k \in Z$. Hence

$$\phi_* = F \bullet X + 2 f^T x = \sum_{k \in P} \lambda_k \left((x^k)^T F x^k + 2 f^T x^k \right).$$

By the definition of ϕ_*, each summand on the right is at least $\lambda_k \phi_*$, and since $\lambda_k > 0$ and $\sum_{k \in P} \lambda_k = 1$, it follows that each $(x^k)^T F x^k + 2 f^T x^k = \phi_*$, as desired. □

A common situation in which the condition $d^T (F \bar{x} + f) = 0$ for all $d \in \mathcal{L}_\infty$ occurs when $\mathcal{L}$ is bounded and consequently $\mathcal{L}_\infty = \{0\}$. Another situation is when all variables x_j involved in $x^T F x + 2 f^T x$ are bounded.

As an example, consider the quadratic equation $x_i x_j = 0$ when $\mathcal{L}$ implies both x_i and x_j are nonnegative and bounded. Then Theorem 8.3 shows that the constraint $X_{ij} = 0$ in $\mathcal{R}'$ captures the complementarity constraint $x_i x_j = 0$. Similarly, the binary condition $x_i \in \{0, 1\}$ is captured by the equations $x_i^2 = x_i$ and $X_{ii} = x_i$ whenever $\mathcal{L}$ implies $0 \le x_i \le 1$.

Theorem 8.3 also gives the following convex-hull result, which is significant because $\mathcal{L}'$ is generally nonconvex.

Corollary 8.2. $\mathrm{clconv}(\mathcal{L}') = \left\{ x : \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{R}' \textit{ for some } X \right\}.$

Said differently, the closed convex hull of $\mathcal{L}$ intersected with the equation $x^T F x + 2 f^T x = \phi_*$ is the projection of $\mathcal{R}'$ onto the coordinates corresponding to x.

Multiple quadratic constraints $\{x^T F^j x + 2 (f^j)^T x = (\phi^j)_*\}$, where

$$(\phi^j)_* := \min\{x^T F^j x + 2 (f^j)^T x : x \in \mathcal{L}\},$$
$$(\phi^j)^* := \max\{x^T F^j x + 2 (f^j)^T x : x \in \mathcal{L}\},$$

may be easily incorporated as long as there exists $\bar{x} \in \mathcal{L}$ satisfying all of the quadratic constraints and each quadratic constraint individually satisfies the assumptions of Theorem 8.3.

8.3 Optimization and Duality Results

In this section, we apply the convex hull results of Sect. 8.2 to the optimization (8.2) and related problems. We then discuss some basic duality results.

8.3.1 Optimization

By standard results, (8.2) may be expressed as

$$\nu_* = \min\left\{\widehat{Q} \bullet Y : Y \in \text{clconv}(\mathcal{L}^1)\right\}, \quad \text{where } \widehat{Q} := \begin{pmatrix} 0 & c^T \\ c & Q \end{pmatrix}.$$

Thus, Theorem 8.1 implies

$$\nu_* = \min\left\{\widehat{Q} \bullet Y : Y \in \mathcal{R}\right\}. \tag{8.11}$$

We formally state the relationship between (8.2) and (8.11) in the following corollary.

Corollary 8.3. *The nonconvex quadratic conic program (8.2) is equivalent to the linear conic program (8.11), i.e.: (i) both share the same optimal value; (ii) if*

$$Y^* = \begin{pmatrix} 1 & (x^*)^T \\ x^* & X^* \end{pmatrix}$$

is optimal for (8.11), then x^* is in the convex hull of optimal solutions for (8.2).

Proof. Item (i) follows from the preceding discussion. To prove (ii), we assume without loss of generality that $\nu_* > -\infty$ and claim $d^T Qd \geq 0$ for all $d \in \mathcal{L}_\infty$. If not, then there exists a nonnegative direction d along which the objective $x^T Qx + 2c^T x$ can be driven to $-\infty$.

Item (ii) is then proved by examining the representation (8.7) for Y^*. In such a case, we must have x^k optimal for (8.2) for all $k \in P$; otherwise, $\widehat{Q} \bullet Y^*$ could not equal ν_*. In fact, we know $(z^k)^T Qz^k = 0$ for all $k \in Z$. Since $x^* = \sum_{k \in P} \lambda_k x^k$, item (ii) follows. □

Note that item (ii) of the corollary does not imply that x^* is itself optimal, just that it is a convex combination of optimal solutions.

In the context of Sect. 8.2.3, we also conclude that (8.2) is equivalent to

$$\min\left\{Q \bullet X + 2c^T X\alpha : X \in \mathcal{R}_0\right\} \tag{8.12}$$

as stated in the following corollary of Theorem 8.2.

Corollary 8.4. *Suppose (8.9) holds, and define α via (8.10). Then (8.12) is equivalent to (8.2), i.e.: (i) both share the same optimal value; (ii) if X^* is optimal for (8.12), then $X^*\alpha$ is in the convex hull of optimal solutions for (8.2).*

A similar results also holds for the situation of quadratic constraints discussed in Sect. 8.2.4 relative to the optimization problems

$$\min\left\{x^T Qx + 2c^T x : \begin{array}{c} Ax = b, x \in \mathcal{K} \\ x^T Fx + 2f^T x = \phi_* \end{array}\right\} \tag{8.13}$$

$$\min\left\{\widehat{Q} \bullet Y : Y \in \mathcal{R}'\right\}. \tag{8.14}$$

Corollary 8.5. *Suppose both ϕ_* and ϕ^* are finite and there exists $\bar{x} \in \mathcal{L}'$ such that $d^T(F\bar{x} + f) = 0$ for all $d \in \mathcal{L}_\infty$. Then (8.14) is equivalent to (8.13), i.e.: (i) both share the same optimal value; (ii) if*

$$Y^* = \begin{pmatrix} 1 & (x^*)^T \\ x^* & X^* \end{pmatrix}$$

is optimal for (8.14), then x^* is in the convex hull of optimal solutions for (8.13).

8.3.2 Duality

We now investigate some basic duality results for the linear conic problem (8.11), which is equivalent to (8.2) via Corollary 8.3. We first prove a technical detail that helps to interpret some of the results.

Proposition 8.4. *Suppose A has full row rank. Then the constraints of problem (8.11) have the full-row-rank property.*

Proof. The full-row-rank property for (8.11) is equivalent to linear independence of the following $2m+1$ matrices ($i = 1,\ldots,m$):

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_i^T \\ a_i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & a_i a_i^T \end{pmatrix}$$

where a_i^T is the i-th row of A. Because the three types of matrices act in different portions of the matrix space, it suffices to show that the three types are separately independent. The first is a singleton; so independence is clear. For the second type, independence holds because A has full row rank. For the third, let d_i be multipliers such that

$$\sum_{i=1}^m d_i a_i a_i^T = 0 \iff A^T DA = 0$$

where D is the diagonal matrix containing d_i. Since A has full row rank, $(AA^T)^{-1}$ exists, and so $AA^T DAA^T = 0 \Leftrightarrow D = 0$, as desired. □

The dual cone of C is

$$\begin{aligned} C^* :=& \{S : S \bullet Y \geq 0 \;\; \forall\, Y \in C\} \\ =& \left\{S : \binom{\zeta}{z}^T S \binom{\zeta}{z} \geq 0 \;\; \forall \binom{\zeta}{z} \in \mathfrak{R}_+ \times \mathcal{K}\right\} \end{aligned}$$

with interior

$$\text{int}(\mathcal{C}^*) := \left\{ S : \binom{\zeta}{z}^T S \binom{\zeta}{z} > 0 \ \ \forall\, 0 \neq \binom{\zeta}{z} \in \Re_+ \times \mathcal{K} \right\}.$$

Note that $\text{int}(\mathcal{C}^*) \neq \emptyset$; for example, any positive definite S is an element of $\text{int}(\mathcal{C}^*)$.

We use $\mathcal{C}^*$ to derive the dual of the formulation (8.11). By standard constructions, the dual is

$$\max \left\{ \lambda + b^T y + (b \circ b)^T w : \widehat{Q} - \begin{pmatrix} \lambda & \frac{1}{2} y^T A \\ \frac{1}{2} A^T y & A^T \text{Diag}(w) A \end{pmatrix} \in \mathcal{C}^* \right\}. \tag{8.15}$$

In general, one needs to verify some constraint qualification in order to guarantee that strong duality holds. As mentioned in the previous subsection, (8.11) never has an interior. This implies that the level sets of (8.15) are unbounded under Proposition 8.4 [2], which includes the optimal solution set (if it exists). If (8.15) has interior, then the optimal value ν_* of (8.11) is attained. One checkable, sufficient condition for (8.15) to have an interior is stated in the following proposition.

Proposition 8.5. *Suppose there exist λ, y, w such that*

$$\widehat{Q} - \begin{pmatrix} \lambda & \frac{1}{2} y^T A \\ \frac{1}{2} A^T y & A^T \text{Diag}(w) A \end{pmatrix} \succ 0.$$

Then (8.15) has interior, and (8.11) attains its optimal value ν_*.

8.4 Working with the Cone $\mathcal{C}$

In this section, we restrict our attention to cones $\mathcal{K}$ that are the Cartesian product of a nonnegative orthant, second-order cones, semidefinite cones, and a Euclidean space.

8.4.1 Basic Results

In general, the cones $\mathcal{C}$ and $\mathcal{C}^*$ are intractable. For example, when $\mathcal{K} = \Re^n_+$, checking $S \in \mathcal{C}^*$ is co-NP complete [27]. On the other hand, some basic results are known or straightforward to prove. We define

$$\mathcal{I} := \{x \in \Re^n : x_1 \geq \|(x_2, \ldots, x_n)\|\} \qquad \text{(second-order or "ice cream")}$$

$$\mathcal{P} := \{X \text{ symmetric matrix} : X \succeq 0\} \qquad \text{(positive semidefinite)}$$

$$\mathcal{N} := \{X \text{ square matrix} : X \geq 0\} \qquad \text{(nonnegative)}$$

In each of the following propositions, $\mathcal{C}$ is the set of $(n+1)\times(n+1)$ matrices, which are completely positive over $\Re_+ \times \mathcal{K}$:

Proposition 8.6. *If $\mathcal{K} = \Re^n_+$, then $\mathcal{C} \subseteq \mathcal{P} \cap \mathcal{N}$. In addition, equality holds if and only if $n \leq 3$.*

Proof. The definition of $X \in \mathcal{C}$ implies $X \in \mathcal{P} \cap \mathcal{N}$. The low-dimension result is due to [26]. □

To keep the dimensions clear, we caution the reader that $\mathcal{P} \cap \mathcal{N}$ consists of size $(n+1)\times(n+1)$ matrices.

Proposition 8.7. *If $\mathcal{K} = \mathcal{I}^n$, then*

$$\mathcal{C} \subseteq \left\{ \begin{pmatrix} \chi & x^T \\ x & X \end{pmatrix} \succeq 0 : \begin{array}{c} x \in \mathcal{I}^n \\ X_{22} + \cdots + X_{nn} \leq X_{11} \end{array} \right\}.$$

Proof. Let $\bar{x} \in \mathcal{I}$. To prove the result, it suffices to show

$$\begin{pmatrix} \chi & x^T \\ x & X \end{pmatrix} := \begin{pmatrix} 1 \\ \bar{x} \end{pmatrix} \begin{pmatrix} 1 \\ \bar{x} \end{pmatrix}^T = \begin{pmatrix} 1 & \bar{x}^T \\ \bar{x} & \bar{x}\bar{x}^T \end{pmatrix}$$

is in the right-hand-side set. Positive semidefiniteness is clear, and so is $x \in \mathcal{I}$. The constraint $X_{22} + \cdots + X_{nn} \leq X_{11}$ is equivalent to $\bar{x}_2^2 + \cdots + \bar{x}_n^2 \leq \bar{x}_1^2$, which is true because $\bar{x} \in \mathcal{I}$. □

In fact, [35] shows that

$$\mathcal{C}_0 := \operatorname{cone}(\{xx^T : x \in \mathcal{I}\}) = \{X \succeq 0 : X_{22} + \cdots + X_{nn} \leq X_{11}\}$$

which can be viewed as a strengthening of Proposition 8.7 in the bottom-right $n \times n$ block.

Proposition 8.8. *If $\mathcal{K} = \Re^n$, then $\mathcal{C} = \mathcal{P}$.*

Proof. $\mathcal{C} \subseteq \mathcal{P}$ is clear. To prove the reverse inclusion, let $Y \in P$ and write a Gram representation $Y = \sum_k y^k (y^k)^T$ for $y^k \in \Re^{n+1}$. Without loss of generality, the first component of each y^k is nonnegative; if not, just negate y^k without affecting $y^k(y^k)^T$. This shows $Y \in \mathcal{C}$. □

In the next subsection, we discuss further results for the case $\mathcal{K} = \Re^n_+$. To our knowledge, however, the above are the only known results for $\mathcal{K}$ involving Euclidean spaces and second-order cones. Nothing is known when $\mathcal{K}$ is a semidefinite cone. In addition, the case when $\mathcal{K}$ is the mixed Cartesian product of such cones has not been studied.

8.4.2 More When $\mathcal{K} = \Re^n_+$

The case of the nonnegative orthant, i.e., when $K = \Re^n_+$ and $\mathcal{C}$ is the cone of completely positive matrices, has received considerable attention in the literature. The recent monograph [4] studies $\mathcal{C}$ from the point of view of linear algebra, and the survey [25] covers $\mathcal{C}^*$ from the point of view of convex analysis. We focus on relatively recent results from the optimization point of view.

As mentioned in the Introduction, [13, 30] discuss a hierarchy of linear- and semidefinite-representable cones approximating $\mathcal{C}^*$ from the inside. More precisely, there exist closed, convex cones $\mathcal{D}^*_r$ $(r = 0,1,2,\ldots)$ such that $\mathcal{D}^*_r \subset \mathcal{D}^*_{r+1}$ for all $r \geq 0$ and $\mathrm{cl}(\cup_r \mathcal{D}^*_r) = \mathcal{C}^*$. The corresponding dual cones $\mathcal{D}_r$ approximate $\mathcal{C}$ from the outside: $\mathcal{D}_r \supset \mathcal{D}_{r+1}$ for all r and $\cap_r \mathcal{D}_r = \mathcal{C}$. Explicit representations of the approximating cones have been worked out in [6, 30]. For example, $\mathcal{D}_0$ is the cone of so-called *doubly nonnegative matrices* – $\mathcal{P} \cap \mathcal{N}$ as introduced in Proposition 8.6. Moreover, using these approximating cones, [6, 22, 31] prove approximation results for several NP-hard problems. Variations of $\mathcal{D}_r$ and $\mathcal{D}^*_r$ have been presented in [31, 37], and adaptive polyhedral approximations similar in spirit have been presented in [8]. Another type of hierarchy is presented in [12].

A recent line of research has examined $\mathcal{C}$ for small values of n. This can shed light on larger completely positive matrices since principal submatrices are completely positive. According to Proposition 8.6, 5×5 is the smallest size for which the doubly nonnegative matrices do *not* capture the completely positive matrices. (This corresponds to $n = 4$ in our notation.) The papers [11, 15] provide closed-form inequalities to separate structured 5×5 matrices in $\mathcal{P} \cap \mathcal{N} \setminus \mathcal{C}$. [12] provides a separation algorithm for 5×5 completely positive matrices, which establishes that 5×5 completely positive matrices are tractable.

Computationally, approaches involving techniques of the two preceding paragraphs have mostly been limited to small or medium sized problems. For the approximating cones $\mathcal{D}_r$ and $\mathcal{D}^*_r$, the size of the description of these cones grows exponentially with r, and even $r = 0$ can present a challenge for off-the-shelf interior-point methods for, say, $n \geq 100$. Working with 5×5 principal submatrices of an $n \times n$ completely positive matrix also presents challenges because there are $O(n^5)$ such submatrices.

As an alternative to interior-point methods, several large-scale algorithms [10, 36, 38] have been used to solve semidefinite programs over $\mathcal{D}_0 = \mathcal{P} \cap \mathcal{N}$. The key idea is to decouple the positive semidefinite constraint of $\mathcal{P}$ from the nonnegativity constraint of $\mathcal{N}$, and then to devise an algorithmic framework that nevertheless handles the decoupling, e.g., a decomposition method. Successful results have been reported up to $n \approx 1{,}000$. Exploiting symmetry [14, 23] is also a promising technique to increase the size of solvable problems.

The authors of [8] also report the success of their adaptive algorithm for solving copositive programs on standard quadratic programs (see Sect. 8.5) up to size $n = 10{,}000$.

8.5 Applications

We close this chapter with a brief discussion of a few specific applications of linear conic programs over C that have appeared in the literature. The applications are given roughly in chronological order.

As far as we are aware, [7] establishes the first copositive representation of an NP-hard problem.

Theorem 8.4. *The* standard quadratic program $\min\{x^T Qx : e^T x = 1, x \geq 0\}$ *is equivalent to the linear conic program* $\min\{Q \bullet X : e^T Xe = 1, X \in C_0\}$*, where* C_0 *is the cone of matrices, which are completely positive over* $\mathcal{K} = \mathfrak{R}^n_+$*, and* $e \in \mathfrak{R}^n$ *is the all-ones vector.*

The paper [13] shows that the NP-hard maximum stable set problem is a completely positive program.

Theorem 8.5. *Let* $G = (V, E)$ *be an undirected graph with vertex set* $V = \{1, \ldots, n\}$ *and edge set* $E \subseteq V \times V$*. The* maximum stable set problem *on* G *is equivalent to the linear conic program*

$$\max\left\{e^T Xe : \begin{array}{c} X_{ij} = 0 \;\forall\, (i, j) \in E \\ \mathrm{trace}(X) = 1, X \in C_0 \end{array}\right\}$$

where C_0 *is the cone of matrices, which are completely positive over* $\mathcal{K} = \mathfrak{R}^n_+$.

The authors also establish an explicit, finite bound on the size r which guarantees that the maximum stable set size is achieved (after rounding down) when $\mathcal{D}_r$ is used as an approximation of the completely positive cone. Later papers [22, 31] improve upon this bound. Related to these results, [24] shows the following:

Theorem 8.6. *The chromatic number* χ *of a graph* G *is the optimal value of a completely positive program.*

Related results can be found in [16].

In the thesis [32] and related papers [33, 34], it is shown that a certain class of quadratic programs over transportation matrices can be represented as completely positive programs. Transportation matrices are element-wise nonnegative with pre-specified row- and column-sums. One example of this is the quadratic assignment problem.

Theorem 8.7. *The quadratic assignment problem can be formulated as a completely positive program.*

We mention that the specific derivation of Theorem 8.7 in [34] is not subsumed by the techniques of this chapter (though the results herein can be used to give a second derivation of the theorem). However, Theorem 8.7 uses the Cauchy–Schwarz inequality just as the proof of Theorem 8.1 does.

The paper [9] contains the result on which this chapter is primarily based.

Theorem 8.8. *Any nonconvex quadratic program having a mix of binary and continuous variables, as well as complementarity constraints on bounded variables, can be formulated as a completely positive program.*

Using similar proof techniques, [28] shows the following:

Theorem 8.9. *Consider a 0-1 integer program with uncertain objective vector, which nevertheless has known first moment vector and second moment matrix. The expected optimal value of the integer program can be formulated as a completely positive program.*

This theorem finds applications in order statistics and project management.

Combining Theorem 8.1 and the low-dimensional case of Proposition 8.6, which establish $\mathcal{C} = \mathcal{P} \cap \mathcal{N}$ for $n \leq 3$, [1] investigates low-dimensional convex hulls.

Theorem 8.10. *Let $\mathcal{K} = \Re^n_+$. For $n \leq 3$,*

$$\operatorname{clconv}(\mathcal{L}^1) = \left\{ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \in \mathcal{P} \cap \mathcal{N} : \begin{array}{c} Ax = b \\ \operatorname{diag}(AXA^T) = b \circ b \end{array} \right\}.$$

For $n \leq 4$, suppose in addition that (8.9) holds, and define α via (8.10). Then

$$\operatorname{clconv}(\mathcal{L}^1) = \left\{ \begin{pmatrix} 1 & \alpha^T X \\ X\alpha & X \end{pmatrix} : \begin{array}{c} AX\alpha = b \\ \operatorname{diag}(AXA^T) = b \circ b \\ \alpha^T X \alpha = 1 \\ X \in \mathcal{P} \cap \mathcal{N} \end{array} \right\}.$$

The authors use this approach, for example, to derive a closed-form representation of

$$\operatorname{clconv}\left(\left\{ \binom{1}{x}\binom{1}{x}^T : 0 \leq x \leq e \right\} \right)$$

where $x \in \Re^2$, which previously had only been partially characterized in the global optimization literature. To achieve the result, the authors add slack variables to form the system $\{(x,s) \geq 0 : x + s = e\}$ and then apply the theorem with dimension $n = 4$.

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