Second-Order-Cone Constraints for Extended Trust-Region Subproblems

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Abstract

The classical trust-region subproblem (TRS) minimizes a nonconvex quadratic objective over the unit ball. In this paper, we consider extensions of TRS having extra constraints. When two parallel cuts are added to TRS, we show that the resulting nonconvex problem has an exact representation as a semidefinite program with additional linear and second-order-cone constraints. For the case where an additional ellipsoidal constraint is added to TRS, resulting in the "two trust-region subproblem" (TTRS), we provide a new relaxation including second-order-cone constraints that strengthens the usual SDP relaxation.

Keywords: trust-region subproblem, second-order cone programming, semidefinite programming, nonconvex quadratic programming.

Mathematics Subject Classification: 90C20, 90C22, 90C25, 90C26, 90C30.

1 Introduction

The classical *trust-region subproblem* (TRS) minimizes a nonconvex quadratic objective over the unit ball: $v(\text{TRS}) := \min\{x^T Qx + c^T x : \|x\| \le 1\}$. TRS is a key subproblem in trust-region methods for nonlinear optimization [6], and several efficient algorithms are available for its solution [8, 11, 16]. The complexity to construct an ϵ-optimal solution of TRS is

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considered in [7, 20]. Although TRS is nonconvex, it is well known that $v(\text{TRS})$ equals the optimal value of the following polynomial-time solvable semidefinite program (SDP) [16]:

$$v(\text{TRS}) = \min \left\{ Q \bullet X + c^T x : \text{trace}(X) \le 1, \; X \succeq xx^T \right\}. \tag{1}$$

Indeed, this equivalence with a convex problem provides one perspective on the tractability of TRS; [18] provides another. For additional background on TRS, we refer the reader to [16, Section 1.1].

The paper [16] also mentions several applications of TRS with additional constraints. For example, [5] presents a sequential quadratic programming method that makes use of the *two trust-region subproblem* (TTRS), which is TRS with an additional strictly convex quadratic constraint. Another variant has a second non-strictly convex quadratic constraint [21]. TRS with extra constraints also arises in the analysis and relaxation of NP-hard combinatorial optimization problems [14], where semidefinite programming plays an important role. Indeed, any *quadratically constrained quadratic program* (QCQP) with at least one strictly convex quadratic constraint can be viewed as an extension of TRS, with a natural semidefinite relaxation. The exact nature of the additional constraints, of course, determines the strength of that relaxation.

For the case where a single linear constraint $a^T x \le u$ is added to TRS—a problem which we denote as TRS1—the paper [19] shows that the problem's optimal value can also be calculated by solving a convex program, in this case an SDP with one added second-order cone (SOC) constraint:

$$\min \left\{ Q \bullet X + c^T x : \begin{array}{c} \|ux - Xa\| \le u - a^T x \\ \text{trace}(X) \le 1, \; X \succeq xx^T \end{array} \right\}. \tag{2}$$

The derivation of the constraint $\|ux - Xa\| \le u - a^T x$ in (2) appears to be unique in the literature. To generate valid linear constraints in (x, X), a common approach involves multiplying together two valid linear constraints in x to create a redundant quadratic constraint, which is then linearized via $X = xx^T$. Such constraints are commonly referred to as "RLT" constraints, after the reformulation-linearization technique of [17]. In contrast, the SOC constraint in (2) is obtained by linearizing the valid quadratic SOC constraint $\|(u - a^T x)x\| = (u - a^T x)\|x\| \le u - a^T x$. We call an SOC constraint obtained in this fashion an "SOC-RLT" constraint. Subsequent to [19], SOC-RLT constraints have also appeared in [21] and [4].

In this paper, we study TRS with extra constraints, and in particular, we are interested in tight SDP relaxations that employ ordinary RLT and SOC-RLT constraints. We focus on four problems, which are organized by the complexity of the geometry of their feasible regions: (i) TRS1, which is TRS with one cut; (ii) TRS2p, which is TRS with two parallel cuts; (iii) TRS2, which is TRS with two general cuts (either intersecting or non-intersecting within the ball $\{x : \|x\| \leq 1\}$); and (iv) TTRS, which is TRS with a second full-dimensional ellipsoid constraint. Note that two parallel cuts can be viewed as a degenerate ellipsoidal constraint. In Section 2, we re-prove the result of [19], which shows that TRS1 is representable as a mixed SOCP/SDP. Our proof of this result introduces important machinery that will be required later in the paper. In Section 3, we show that TRS2p also has a representation as an SDP with three added constraints: a single RLT constraint and two SOC-RLT constraints. Then in Section 4, we provide an example to show that the analogous SDP is not tight for TRS2, at least when the two cuts are intersecting. Finally, in Section 5, we provide a new family of polynomial-time separable SOC-RLT constraints for TTRS (which in fact apply to any QCQP with two or more convex quadratic constraints). We show that the use of these constraints resolves several known examples where there is a gap between the solution value of TTRS and its standard SDP relaxation. At the same time, we show instances of TTRS where the addition of our new SOC-RLT constraints to the SDP relaxation closes most—but not all—of the gap.

An important implication of our results is that the computational complexity of solving an extended trust-region problem is highly dependent on the geometry of the feasible set. For example, if the feasible set is a ball cut by two parallel half spaces (TRS2p), then the problem is polynomial-time solvable. On the other hand, if the two half-spaces are not parallel and furthermore intersect within the ball (the intersecting case of TRS2), the complexity is unknown. A more complicated geometry (e.g., TTRS) appears to be even more difficult.

Both TRS2p and TTRS were considered in [21], where the authors establish "trajectory following" procedures that solve these problems. (As far as we are aware, [21] is the only paper in the literature to study TRS2p formally, and in fact this paper motivated our study here.) The procedure for TRS2p is actually polynomial-time but requires the consideration of two separate cases. The procedure for TTRS is not known to be polynomial but appears to be quite efficient in practice. The authors of [21] asked whether there are exact convex formulations of TRS2p and TTRS. This paper answers the question affirmatively for TRS2p and also provides a tighter—but still not exact—relaxation scheme for TTRS. The computational complexity of TTRS remains an open question.

We point out that TTRS is a well studied problem. A classic reference establishing optimality conditions is Peng and Yuan [13], and this paper cites many references from the 1990s. More recently, Beck and Eldar [2] and Ai and Zhang [1] also consider TTRS (and generalizations of it), and Ai-Zhang give a thorough background on TTRS up until 2009. Each of these papers is concerned with necessary and/or sufficient conditions describing when the standard SDP relaxation of TTRS is tight. Of course, these conditions do not apply to all instances. The approach that we take in this paper is to strengthen the SDP relaxation generally, and we demonstrate the success of our approach computationally.

Notation. For symmetric matrices X and Y, $Y \succeq X$ denotes that $Y - X$ is positive semidefinite, and $X \bullet Y$ is the matrix inner product $X \bullet Y = \text{trace}(XY)$. The operator $\text{diag}(X)$ returns the diagonal of X as a vector, and $X_{\cdot j}$ denotes the jth column of X. We use e to denote a vector of suitable dimension with each component equal to one, and e_1 to denote a vector whose first component is one and whose remaining components are zero. For a vector u we use $(1; u)$ to denote the vector $\binom{1}{u}$.

2 TRS with one cut

In this section, we study the problem TRS1, which is the ordinary TRS with an additional linear constraint $a^T x \le u$. In particular, we re-establish the result of Sturm and Zhang [19] that $v(\text{TRS1})$ can be represented as the optimal value of a mixed SOCP/SDP. Our intent is to introduce machinery that will be needed in Section 3 and to provide an alternative proof of the result that may be of independent interest. (Beck and Eldar [2] also studied TRS1 but only discussed conditions under which the basic SDP relaxation is tight.)

In order to simplify notation (here and in Section 3), we employ an orthogonal transformation of $\Re^n$ to put TRS1 in the form

$$v(\text{TRS1}) := \min \left\{ x^T Q x + c^T x : x_1 \le u, \ \|x\| \le 1 \right\}.$$

For the problem in this form, the corresponding SOCP/SDP relaxation is

$$r(\text{TRS1}) := \min \left\{ Q \bullet X + c^T x : \begin{array}{c} \|ux - X_{\cdot 1}\| \le u - x_1 \\ \text{trace}(X) \le 1, \ X \succeq xx^T \end{array} \right\}. \tag{3}$$

By the following proposition, (3) is equivalent to the SOCP/SDP relaxation (2) that would be obtained without first performing the orthogonal transformation.

Proposition 1. *Let P be an orthogonal matrix such that $Pa = e_1$. Then the feasible sets of (3) and*

$$\min \left\{ (PQP^T) \bullet Z + (P^T c)^T z : \begin{array}{c} \|uz - Za\| \le u - a^T z \\ \text{trace}(Z) \le 1, \ Z \succeq zz^T \end{array} \right\}$$

are in bijective correspondence via the invertible mapping $(x, X) \leftrightarrow (Pz, PZP^T)$*. In addition, corresponding points* (x, X) *and* (z, Z) *share the same objective value, so that the optimal values are equal.*

Proof. Let (z, Z) be feasible, and define $(x, X) = (Pz, PZP^T)$. Then

$$\begin{aligned} \|ux - X_{\cdot 1}\| &= \|uPz - PZP^T e_1\| = \|P(uz - Za)\| = \|uz - Za\| \\ &\le u - a^T z = u - e_1^T Pz = u - x_1. \end{aligned}$$

Also, $\text{trace}(X) = \text{trace}(PZP^T) = \text{trace}(Z) \le 1$ and $X = PZP^T \succeq Pzz^T P^T = xx^T$. So (x, X) is feasible. Given feasible (x, X), a similar argument shows that $(z, Z) = (P^T x, P^T XP)$ is feasible. Moreover $Q \bullet X + c^T x = Q \bullet (PZP^T) + c^T(Pz) = (P^T QP) \bullet Z + (P^T c)z$, and therefore corresponding objective values are equal. □

We would like to prove $r(\text{TRS1}) = v(\text{TRS1})$. We will accomplish this by showing that every extreme point (x, X) of the feasible set of (3) satisfies $\text{rank}[Y(x, X)] = 1$, where

$$Y(x, X) := \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}.$$

Note that $\text{rank}[Y(x, X)] = 1 \iff X = xx^T$. Our proof relies in part on the following result for extreme points of semidefinite constraint systems due to Pataki [12]:

Proposition 2. *Consider an SDP feasible set in block standard form:* $F := \{X^j \succeq 0,\ j = 1, \ldots, p : \sum_{j=1}^p A_i^j \bullet X^j = b_i,\ i = 1, \ldots, m\}$*. Let* $(X^1, \ldots, X^p)$ *be an extreme point of* F*, and define* $r_j := \text{rank}(X^j)$*. Then* $\sum_{j=1}^p r_j(r_j + 1) \le 2m$*.*

Using Proposition 2, we can give a very short proof of the following lemma that shows that the SDP representation of TRS in (1) is correct.

Lemma 1. *Suppose that* (x, X) *is an extreme point of the convex set* $\{(x, X) : \text{trace}(X) \le \theta,\ X \succeq xx^T\}$ *for some* $\theta > 0$*. Then* $X = xx^T$*.*

Proof. The given convex set can be expressed in the form of Proposition 2 as

$$F := \left\{ Y = \begin{pmatrix} \chi & x^T \\ x & X \end{pmatrix} \succeq 0,\ s \ge 0 : \chi = 1,\ \text{trace}(X) + s = \theta \right\},$$

and clearly (x, X) is extreme if and only if the corresponding (Y, s) is extreme. Proposition 2 then implies that if (Y, s) is extreme in F, $r_Y(r_Y + 1) + r_s(r_s + 1) \le 4$, where $r_Y = \text{rank}(Y)$ and $r_s = \text{rank}(s)$. Then $r_Y \le 1$, and since $Y \ne 0$ it must be that $r_Y = 1$, implying $X = xx^T$. □

To analyze TRS1 we need one additional result, which will also be used in Section 3.

Lemma 2. *Suppose that $X \succeq xx^T$ and $X_{11} = x_1^2$. Then $X_{\cdot 1} = x_1 x$. If in addition (x, X) is an extreme point of the feasible set of (3), then $X = xx^T$.*

Proof. Since $Y(x, X) \succeq 0$, for all $j = 2, \ldots, n$,

$$\det \begin{pmatrix} 1 & x_1 & x_j \\ x_1 & X_{11} & X_{j1} \\ x_j & X_{j1} & X_{jj} \end{pmatrix} = (X_{11} - x_1^2)(X_{jj} - x_j^2) - (X_{j1} - x_1 x_j)^2 \geq 0. \tag{4}$$

Hence, $X_{11} = x_1^2$ implies $X_{\cdot 1} = x_1 x$.

Now assume that (x, X) is an extreme point of (3). Let $\bar{x}$ denote the last $n-1$ components of x and $\bar{X}$ denote the bottom-right $(n-1)\times(n-1)$ principal submatrix of X. If $x_1 = X_{11} = 1$, then $\bar{x} = 0$ and $\bar{X} = 0$, so $X = e_1 e_1^T = xx^T$. If $x_1 < 1$, then $(\bar{x}, \bar{X})$ is feasible for the lower-dimensional set $F(x_1) := \{(w, W) : \text{trace}(W) \leq 1 - x_1^2, W \succeq ww^T\}$. In addition, any $(w, W) \in F(x_1)$ can be used to construct a feasible point of (3) via the linear mapping

$$\begin{pmatrix} 1 & w^T \\ w & W \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ x_1 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 1 & w \\ w & W \end{pmatrix} \begin{pmatrix} 1 & x_1 & 0 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} 1 & x_1 & w^T \\ x_1 & x_1^2 & x_1 w^T \\ w & x_1 w & W \end{pmatrix}.$$

In particular, (x, X) is the image of $(\bar{x}, \bar{X})$ under this mapping. As a result, since (x, X) is extreme in (3), $(\bar{x}, \bar{X})$ must be extreme in $F(x_1)$. Then Lemma 1 implies that every extreme point of $F(x_1)$ has $W = ww^T$, which implies $X = xx^T$. □

We are now ready to prove that every extreme point (x, X) of (3) satisfies $X = xx^T$. We employ a proof technique that takes a general (x, X) that is feasible for (3) and writes $Y(x, X)$ as a structured convex combination of other feasible points. The structure is specific enough so that, when one also assumes (x, X) to be extreme, then one can prove $\text{rank}(Y(x, X)) = 1$ as desired. However, we caution the reader that the structure of the convex combination is not particularly intuitive, though we have tried to simplify the presentation as much as possible. A similar approach is used throughout Section 3.

Theorem 1. *Every extreme point (x, X) of the feasible set of (3) satisfies $X = xx^T$.*

Proof. Let (x, X) be an extreme point of (3). We break the proof into the following cases: $x_1 = u$; $x_1 < u$ and $\|ux - X_{\cdot 1}\| = u - x_1$; or $x_1 < u$ and $\|ux - X_{\cdot 1}\| < u - x_1$.

Assume first that $x_1 = u$, in which case $X_{\cdot 1} = ux$. Then $X_{11} = ux_1 = x_1^2$, and by Lemma 2, $X = xx^T$.

Next assume that $x_1 < u$ and $\|ux - X_{\cdot 1}\| < u - x_1$. Since the second-order-cone constraint is inactive, (x, X) is also an extreme point of $\{(x, X) : \text{trace}(X) \le 1, X \succeq xx^T\}$, and so $X = xx^T$ holds by Lemma 1.

Finally, assume that $x_1 < u$ and $\|ux - X_{\cdot 1}\| = u - x_1$. Then $z := (u - x_1)^{-1}(ux - X_{\cdot 1})$ satisfies $\|z\| = 1$ and

$$z_1 = \frac{ux_1 - X_{11}}{u - x_1} \le \frac{ux_1 - x_1^2}{u - x_1} = x_1 < u,$$

hence (z, zz^T) is feasible for (3). For $\epsilon > 0$, consider the rank-1 shift

$$Y_\epsilon := \begin{pmatrix} \chi_\epsilon & x_\epsilon^T \\ x_\epsilon & X_\epsilon \end{pmatrix} := Y - \epsilon(u - x_1)^2 yy^T,$$

where

$$y = \begin{pmatrix} 1 \\ z \end{pmatrix} = (u - x_1)^{-1} \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \begin{pmatrix} u \\ -e_1 \end{pmatrix} = (u - x_1)^{-1} Y \begin{pmatrix} u \\ -e_1 \end{pmatrix}.$$

Because $y \in \text{Range}(Y)$ and $Y \succeq 0$, we know $Y_\epsilon \succeq 0$ for sufficiently small $\epsilon > 0$. Furthermore, $\chi_\epsilon = 1 - \epsilon(u - x_1)^2$,

$$\begin{aligned} \text{trace}(X_\epsilon) &= \text{trace}(X - \epsilon(u - x_1)^2 zz^T) = \text{trace}(X) - \epsilon(u - x_1)^2 \\ &\le 1 - \epsilon(u - x_1)^2 = \chi_\epsilon, \end{aligned}$$

and

$$\begin{aligned} \begin{pmatrix} u\chi_\epsilon - [x_\epsilon]_1 \\ ux_\epsilon - [X_\epsilon]_{\cdot 1} \end{pmatrix} &= Y_\epsilon \begin{pmatrix} u \\ -e_1 \end{pmatrix} = Y \begin{pmatrix} u \\ -e_1 \end{pmatrix} - \epsilon(u - x_1)^2 yy^T \begin{pmatrix} u \\ -e_1 \end{pmatrix} \\ &= (u - x_1)y - \epsilon(u - x_1)^2(u - z_1)y \\ &= (u - x_1)\left[1 - \epsilon(u - x_1)(u - z_1)\right] y, \end{aligned}$$

which implies $\|ux_\epsilon - [X_\epsilon]_{\cdot 1}\| \le u\chi_\epsilon - [x_\epsilon]_1$ for sufficiently small $\epsilon > 0$ since $y \ne 0$ is in the second-order cone. So $\chi_\epsilon^{-1}(x_\epsilon, X_\epsilon)$ is feasible for (3) for sufficiently small $\epsilon > 0$, and the equation $Y = \chi_\epsilon\,(\chi_\epsilon^{-1} Y_\epsilon) + (1 - \chi_\epsilon) yy^T$ shows that Y is a nontrivial convex combination of feasible points. Since Y is extreme, $\chi_\epsilon^{-1} Y_\epsilon$ must equal yy^T, and $\text{rank}(Y) = 1$. □

Corollary 1. $r(\text{TRS1}) = v(\text{TRS1})$.

3 TRS with two parallel cuts

In this section, we study the case of TRS constrained by two parallel linear inequality constraints $l \le a^T x \le u$, a problem which we call TRS2p. We assume throughout that $l < u$, since otherwise the problem is trivially equivalent to an instance of TRS of dimension $n-1$. By an appropriate orthogonal transformation, the problem may be stated as

$$v(\text{TRS2p}) := \min \left\{ x^T Q x + c^T x : l \le x_1 \le u,\ \|x\| \le 1 \right\}.$$

We will consider the relaxation

$$r(\text{TRS2p}) := \min \left\{ Q \bullet X + c^T x : \begin{array}{c} X_{11} + lu \le (l+u)x_1 \\ \|X_{\cdot 1} - lx\| \le x_1 - l \\ \|ux - X_{\cdot 1}\| \le u - x_1 \\ \operatorname{trace}(X) \le 1,\ X \succeq xx^T \end{array} \right\}. \tag{5}$$

Compared to (3), the relaxation (5) contains the second SOC-RLT constraint coming from $(x_1 - l)\|x\| \le x_1 - l$ as well as the ordinary RLT constraint coming from $(u - x_1)(x_1 - l) \ge 0$. As in the case of TRS1, the relaxation is equivalent to the one that would be obtained without first using the orthogonal transformation.

Similar to Section 2, we will show $r(\text{TRS2p}) = v(\text{TRS2p})$ by demonstrating that every extreme point (x, X) of the feasible set of (5) satisfies $X = xx^T$. We need several lemmas to accomplish this. First, a result analogous to Lemma 2 also holds for (5).

Lemma 3. *Suppose (x, X) is an extreme point of the feasible set of (5) with $X_{11} = x_1^2$. Then $X = xx^T$.*

Proof. The proof for Lemma 2 holds. One simply notes that if $W \succeq ww^T$ and $\operatorname{trace}(W) \le 1 - x_1^2$, then the map

$$\begin{pmatrix} 1 & w^T \\ w & W \end{pmatrix} \to \begin{pmatrix} 1 & x_1 & w^T \\ x_1 & x_1^2 & x_1 w^T \\ w & x_1 w & W \end{pmatrix}$$

results in $Y(x, X)$ feasible for (5). □

Lemma 4. *Let (x, X) be feasible for (5) with $X_{11} > x_1^2$ and $X_{11} + lu = (l+u)x_1$, which ensures $l < x_1 < u$. Then there are $z^1 \neq z^2$, both feasible for* TRS2p, *$0 < \lambda < 1$, and*

$W \in \Re^{(n-1)\times(n-1)}$ *with* $\operatorname{diag}(W) \geq 0$ *such that* $Y(x,X) = (1-\lambda)Y^1 + \lambda Y^2$*, where*

$$Y^1 := \begin{pmatrix} 1 \\ z^1 \end{pmatrix}\begin{pmatrix} 1 \\ z^1 \end{pmatrix}^T + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & W \end{pmatrix}, \qquad Y^2 := \begin{pmatrix} 1 \\ z^2 \end{pmatrix}\begin{pmatrix} 1 \\ z^2 \end{pmatrix}^T.$$

Moreover $\operatorname{rank}[Y(x,X)] \geq 2$*, and* $W = 0$ *if* $\operatorname{rank}[Y(x,X)] = 2$*.*

Proof. The strict inequalities $l < x_1 < u$ can be proved directly from $x_1^2 < X_{11} = (l+u)x_1 - lu$. Define

$$\begin{aligned} z^1 &:= \frac{u - x_1}{X_{11} - x_1^2}\left(X_{\cdot 1} - \left(\frac{ux_1 - X_{11}}{u - x_1}\right)x\right) \\ z^2 &:= \frac{1}{u - x_1}\left(ux - X_{\cdot 1}\right) \\ \lambda &:= \frac{(u - x_1)^2}{(u - x_1)^2 + (X_{11} - x_1^2)}. \end{aligned}$$

We first prove the existence of W such that $Y(x,X) = (1-\lambda)Y^1 + \lambda Y^2$. From the definitions, one can verify algebraically that $x = (1-\lambda)z^1 + \lambda z^2$ and $X_{\cdot 1} = (1-\lambda)z_1^1 z^1 + \lambda z_1^2 z^2$ (see the Appendix for details). In addition, using (4) one can also verify

$$\left(X_{11} - x_1^2\right)\left(X_{jj} - (1-\lambda)(z_j^1)^2 - \lambda(z_j^2)^2\right) \geq 0$$

for all $j \geq 2$. This proves the existence of W with $\operatorname{diag}(W) \geq 0$, as claimed. (Note that the proof to this point does not use the assumption $X_{11} + lu = (l+u)x_1$. This is important because we will need the same result below in Lemma 5 without this assumption.)

Next we show that $z^1 \neq z^2$ are both feasible for TRS2p. Note that $X_{11} + lu = (l+u)x_1$ implies that

$$\begin{aligned} z^1 &= \frac{u - x_1}{(u - x_1)(x_1 - l)}\left(X_{\cdot 1} - \left(\frac{l(u - x_1)}{u - x_1}\right)x\right) \\ &= \frac{1}{x_1 - l}\left(X_{\cdot 1} - lx\right). \end{aligned}$$

Then $X_{11} - lx_1 = ux_1 - lu$ implies that $z_1^1 = u$, and $\|z^1\| \leq 1$ follows from $\|X_{\cdot 1} - lx\| \leq x_1 - l$. The fact that $\|z^2\| \leq 1$ is immediate from $\|ux - X_{\cdot 1}\| \leq u - x_1$. Moreover

$$z_1^2 = \frac{ux_1 - X_{11}}{u - x_1} = \frac{lu - lx_1}{u - x_1} = l,$$

because $X_{11} + lu = (l+u)x_1$. Therefore $z_1^2 < u$, which shows that $z^1 \neq z^2$.

Finally we prove that $\text{rank}[Y(x,X)] \ge 2$, and $W = 0$ if $\text{rank}[Y(x,X)] = 2$. Note that the representation $Y(x,X) = (1-\lambda)Y^1 + \lambda Y^2$ shows that the first two columns of $Y(x,X)$ are linear combinations of $(1; z^1)$ and $(1; z^2)$ and these columns are not proportional to one another because $z_1^1 \ne z_1^2$. Therefore $\text{rank}[Y(x,X)] \ge 2$. In addition, if $\text{rank}[Y(x,X)] = 2$ and $W \ne 0$, then any nonzero column of

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & W \end{pmatrix}$$

must be a linear combination of $(1; z^1)$ and $(1; z^2)$, which is clearly impossible. □

Lemma 5. *Let (x,X) be feasible for (5) with $l < x_1 < u$, $X_{11} > x_1^2$, and either $\|ux - X_{\cdot 1}\| = u - x_1$ or $\|X_{\cdot 1} - lx\| = x_1 - l$. Then all conclusions of Lemma 4 hold, and in addition Y^1 satisfies all constraints of (5) except possibly positive semidefiniteness.*

Proof. We give the proof for the case of $\|ux - X_{\cdot 1}\| = u - x_1$; the case of $\|X_{\cdot 1} - lx\| = x_1 - l$ is similar and is omitted. Let z^1, z^2 and λ be exactly as in the proof of Lemma 4. Then $Y = (1-\lambda)Y^1 + \lambda Y^2$ for W with $\text{diag}(W) \ge 0$, exactly as before.

We next show that $z^1 \ne z^2$ are both feasible for TRS2p. We have $\|z^2\| = 1$,

$$z_1^2 = \frac{ux_1 - X_{11}}{u - x_1} \ge \frac{lu - lx_1}{u - x_1} = l,$$

because $X_{11} + lu \le (l+u)x_1$ and

$$z_1^2 = \frac{ux_1 - X_{11}}{u - x_1} < \frac{ux_1 - x_1^2}{u - x_1} = x_1 < u,$$

because $x_1^2 < X_{11}$. Now consider z^1. First, note that

$$z_1^1 = \frac{u - x_1}{X_{11} - x_1^2}\left(X_{11} - \left(\frac{ux_1 - X_{11}}{u - x_1} \right) x_1 \right) = u.$$

In addition

$$\begin{aligned} 1 \ge \text{trace}(X) &= (1-\lambda)\left(\|z^1\|^2 + \text{trace}(W)\right) + \lambda\|z^2\|^2 \\ &= (1-\lambda)\left(\|z^1\|^2 + \text{trace}(W)\right) + \lambda, \end{aligned} \tag{6}$$

which shows that $\|z^1\| \le 1$. Thus $z^1 \ne z^2$ are both feasible for TRS2p, as claimed. The argument that $\text{rank}[Y(x,X)] \ge 2$, and $W = 0$ if $\text{rank}[Y(x,X)] = 2$, is identical to that used in the proof of Lemma 4.

Finally we prove that Y^1 satisfies all constraints of (5) except possibly $Y^1 \succeq 0$. Given the structure of Y^1, the constraints of (5) are $Y^1 \succeq 0$, $\|z^1\|^2 + \text{trace}(W) \leq 1$, $(z_1^1)^2 + lu \leq (l+u)z_1^1$, $\|z_1^1 z^1 - lz^1\| \leq z_1^1 - l$, and $\|uz^1 - z_1^1 z^1\| \leq u - z_1^1$. The last three are satisfied because $\|z^1\| \leq 1$ and $z_1^1 = u$, while $\|z^1\|^2 + \text{trace}(W) \leq 1$ follows from (6). □

Theorem 2. *Every extreme point (x, X) of the feasible set of (5) satisfies $X = xx^T$.*

Proof. Let (x, X) be an extreme point of (5). We break the proof into the following cases:

(i) $x_1 = l$, $x_1 = u$, or $X_{11} = x_1^2$;

(ii) $l < x_1 < u$, $X_{11} > x_1^2$, and $\|X_{\cdot 1} - lx\| = x_1 - l$ or $\|ux - X_{\cdot 1}\| = u - x_1$;

(iii) $l < x_1 < u$, $X_{11} > x_1^2$, $\|X_{\cdot 1} - lx\| < x_1 - l$, and $\|ux - X_{\cdot 1}\| < u - x_1$.

Case (i) follows by Lemma 3 because either of the conditions $x_1 = l$ and $x_1 = u$ imply $X_{11} = x_1^2$.

Now consider case (iii). Since both second-order-cone constraints are inactive, (x, X) is also an extreme point of

$$\left\{ (x, X) : \begin{array}{c} X_{11} + lu \leq (l+u)x_1 \\ \text{trace}(X) \leq 1, \ X \succeq xx^T \end{array} \right\}, \tag{7}$$

which is equivalent to the following block SDP system in standard form:

$$F := \left\{ Y = \begin{pmatrix} \chi & x^T \\ x & X \end{pmatrix} \succeq 0, \ s, t \geq 0 : \begin{array}{c} X_{11} + lu + s = (l+u)x_1 \\ \chi = 1, \ \text{trace}(X) + t = 1 \end{array} \right\}.$$

Moreover, the extreme points of F and (7) are clearly in bijective correspondence, so (x, X) corresponds to an extreme point $(Y(x, X), s, t)$ of F. Also define $r_Y := \text{rank}(Y)$, $r_s := \text{rank}(s)$, and $r_t := \text{rank}(t)$. Then Proposition 2 implies

$$r_Y(r_Y + 1) + r_s(r_s + 1) + r_t(r_t + 1) \leq 6,$$

in which case $r_Y \leq 2$, and since $Y \neq 0$, $r_Y = 1$ or $r_Y = 2$. If $r_Y = 2$, then $X_{11} + lu = (l+u)x_1$ and $\text{trace}(X) = 1$, but then Lemma 4 shows that Y is a non-trivial convex combination of distinct points in (5), which contradicts the assumption that (x, X) is extreme. So in fact $r_Y = 1$.

Finally, consider case (ii). As in the proof of Lemma 5, we give a detailed proof assuming that $\|ux - X_{\cdot 1}\| = u - x_1$; the analysis for the case of $\|X_{\cdot 1} - lx\| = x_1 - l$ is similar and is omitted. Lemma 5 expresses $Y = (1 - \lambda)Y^1 + \lambda Y^2$ with $0 < \lambda < 1$. Furthermore, Y^2 is feasible for (5) and Y^1 is nearly feasible. For $\epsilon \ge 0$, define

$$Y^1(\epsilon) := (1-\lambda+\epsilon)Y^1 + (\lambda-\epsilon)Y^2,$$

and note $Y^1(0) = Y \succeq 0$. Clearly, Y is also a convex combination of $Y^1(\epsilon)$ and Y^2, where $Y^1(\epsilon)$ satisfies all constraints of (5) except possibly the semidefiniteness condition.

We next prove that, for small $\epsilon > 0$, $Y^1(\epsilon)$ is also positive semidefinite, i.e., $Y^1(\epsilon)$ is feasible for (5). First note that $Y^1(\epsilon) \succeq 0$ if and only if $\tilde{Y}(\epsilon) := (1-\lambda)(1-\lambda+\epsilon)^{-1}Y(\epsilon) \succeq 0$. We have

$$\tilde{Y}(\epsilon) = (1-\lambda)Y^1 + \frac{(1-\lambda)(\lambda-\epsilon)}{1-\lambda+\epsilon}Y^2 = Y + f(\epsilon)\begin{pmatrix}1\\ z^2\end{pmatrix}\begin{pmatrix}1\\ z^2\end{pmatrix}^T,$$

where

$$f(\epsilon) := \frac{(1-\lambda)(\lambda-\epsilon)}{1-\lambda+\epsilon} - \lambda.$$

It holds that $f(0) = 0$ and $f'(\epsilon) = -(1-\lambda)/(1-\lambda+\epsilon)^2 < 0$. So, for small $\epsilon > 0$, $\tilde{Y}(\epsilon)$ equals $Y - \delta(1; z^2)(1; z^2)^T$ for small $\delta > 0$. Since

$$(u - x_1)\begin{pmatrix}1\\ z^2\end{pmatrix} = \begin{pmatrix}u - x_1\\ ux - X_{\cdot 1}\end{pmatrix} = u\begin{pmatrix}1\\ x\end{pmatrix} - \begin{pmatrix}x_1\\ X_{\cdot 1}\end{pmatrix} \in \text{Range}(Y),$$

it follows that $\tilde{Y}(\epsilon) \succeq 0$ for sufficiently small $\epsilon > 0$. Then Y is a nontrivial convex combination of $Y^1(\epsilon)$ and Y^2, both of which are feasible for (5). Since Y is extreme, $Y^1(\epsilon) = Y^2$, and $\text{rank}(Y) = 1$. □

Corollary 2. $r(\text{TRS2p}) = v(\text{TRS2p})$.

4 TRS with two general cuts

The previous section has shown that problem TRS2p, the TRS with two parallel cuts, has an exact representation as the mixed SOCP/SDP (5). An analogous relaxation can be easily developed for the problem TRS2, the TRS with two general cuts. In this short section, we discuss whether this relaxation is tight for TRS2.

To simplify notation, by employing an orthogonal transformation, we may assume without loss of generality that an instance of TRS2 has the form

$$v(\text{TRS2}) := \min\left\{x^T Q x + c^T x : l \le x_1 + \varepsilon x_2,\ x_1 \le u,\ \|x\| \le 1\right\}$$

for some scalar ε. Note that $\varepsilon = 0$ gives parallel cuts, so ε controls the slope of the first cut and consequently whether the two cuts intersect inside the ball $\{x : \|x\| \leq 1\}$. If the cuts do intersect inside, we say the instance of TRS2 is *intersecting*; if not, then the instance is *non-intersecting*. The corresponding relaxation is

$$r(\text{TRS2}) := \min \left\{ Q \bullet X + c^T x : \begin{array}{c} X_{11} + lu + \varepsilon X_{21} \leq (l+u)x_1 + \varepsilon u x_2 \\ \|X_{\cdot 1} - lx + \varepsilon X_{\cdot 2}\| \leq x_1 - l + \varepsilon x_2 \\ \|ux - X_{\cdot 1}\| \leq u - x_1 \\ \text{trace}(X) \leq 1, \ X \succeq xx^T \end{array} \right\}. \tag{8}$$

Consider the non-intersecting case of TRS2. We have not been able to prove whether $v(\text{TRS2})$ equals $r(\text{TRS2})$ in this case; in particular, the proofs of Section 3 for TRS2p do not seem to carry over directly. On the other hand, we have tested many random instances of TRS2 with non-intersecting cuts in low dimensions ($n \leq 10$) and have always found empirically that $r(\text{TRS2})$ does equal $v(\text{TRS2})$. So we conjecture that the two values are equal when the cuts do not intersect.

For the intersecting case, we have found the following example for which $v(\text{TRS2}) < r(\text{TRS2})$. Let $n = 3$, and define an instance of TRS2 with

$$Q = \begin{pmatrix} 2 & 3 & 12 \\ 3 & -19 & 6 \\ 12 & 6 & 0 \end{pmatrix}, \ c = \begin{pmatrix} 14 \\ 14 \\ 9 \end{pmatrix}, \ l = -\tfrac{1}{2}, \ u = 0, \ \varepsilon = \tfrac{5}{4}.$$

Using the global optimization software Couenne [3], one can verify that $v(\text{TRS2}) \approx -12.9419$ with $x^* \approx (-0.8529, -0.2941, 0.4313)^T$. In contrast, $r(\text{TRS2}) \approx -13.8410$ with optimal

$$\bar{x} \approx \begin{pmatrix} -0.3552 \\ 0.3881 \\ -0.2119 \end{pmatrix}, \ \bar{X} \approx \begin{pmatrix} 0.2595 & -0.2248 & -0.0913 \\ -0.2248 & 0.4495 & -0.0694 \\ -0.0913 & -0.0694 & 0.2911 \end{pmatrix}.$$

In particular $Y(\bar{x}, \bar{X})$ has numeral rank 3. The relative gap for this instance is thus about 7%.

Our experiences with TRS2 thus suggest two interesting avenues of further research: (i) either prove or disprove that $v(\text{TRS2})$ equals $r(\text{TRS2})$ when the cuts are non-intersecting; (ii) develop ways to strengthen the SOCP/SDP relaxation in order to close the gap when they are intersecting.

5 Two trust regions

In this section, we consider the so-called *two trust region subproblem* (TTRS), which is the regular TRS with an additional full-dimensional ellipsoidal constraint:

$$v(\text{TTRS}) = \min \left\{ x^T Q x + c^T x : \begin{array}{c} \|x\| \le 1 \\ \|H^{1/2}(x-h)\| \le 1 \end{array} \right\},$$

where $H \succ 0$ and $h \in \Re^n$ is the center of the second ellipsoid. We remark that TTRS is in fact just a special case of the generalized trust-region subproblem for nonlinear constrained optimization introduced and studied in [5, 15], where the more general constraint $\|A^T x - b\| \le \xi$ is added to TRS for some $A \in \Re^{m \times n}$. In this sense, TTRS and TRS2p represent two extreme cases with $\|A^T x - b\| \le \xi$ defining a full dimensional ellipsoid and a highly degenerate ellipsoid.

The standard SDP relaxation of TTRS is

$$r(\text{TTRS}) = \min \left\{ Q \bullet X + c^T x : \begin{array}{c} H \bullet X - 2h^T H x + h^T H h \le 1 \\ \text{trace}(X) \le 1, \; X \succeq xx^T \end{array} \right\}.$$

It is not immediately clear how to strengthen this SDP relaxation. For example, there are no explicit linear inequality constraints from which to derive SOC-RLT constraints.

We suggest to derive SOC-RLT constraints from supporting hyperplanes of the ball $B := \{x : \|x\| \le 1\}$. Let $\text{bd}(B) := \{x : \|x\| = 1\}$ denote the boundary of B. Given any vector $a \in \text{bd}(B)$, the inequality $a^T x \le 1$ supports B at a, and so the SOC-RLT constraint $\|H^{1/2}(x - Xa - (1 - a^T x)h)\| \le 1 - a^T x$ strengthens the SDP relaxation. This suggests the following improved relaxation:

$$r^+(\text{TTRS}) = \min \left\{ Q \bullet X + c^T x : \begin{array}{c} H \bullet X - 2h^T H x + h^T H h \le 1 \\ \text{trace}(X) \le 1, \; X \succeq xx^T \\ (x, X) \in \mathcal{H} \end{array} \right\}$$

where

$$\mathcal{H} := \left\{ (x, X) : \|H^{1/2}(x - Xa - (1 - a^T x)h)\| \le 1 - a^T x \;\; \forall \, a \in \text{bd}(B) \right\}.$$

The set $\mathcal{H}$ can also be interpreted as enforcing all of the ordinary RLT constraints coming from pairs $a^T x \le 1$ and $b^T x \le \gamma$ of supporting hyperplanes of B and the second ellipsoid $\{x : \|H^{1/2}(x - h)\| \le 1\}$, respectively. Fixing $a^T x \le 1$, all of these ordinary RLT constraints can be conveniently combined into the single SOC-RLT constraint presented. Then $\mathcal{H}$ is derived by varying a. In analogy with $\mathcal{H}$, one could also generate another set from SOC-RLT cuts, say, $\mathcal{H}'$ using the supporting hyperplanes of the second ellipsoid. However, $\mathcal{H}'$ could also be interpreted as enforcing all of the ordinary RLT constraints between the two ellipsoids, and so one can show formally that $\mathcal{H}' = \mathcal{H}$ with no further strengthening of r^+(TTRS).

The semi-infinite SOCP/SDP defining r^+(TTRS) does not appear to have an equivalent finite representation that can be solved directly. Indeed, its bilinear nature involving the terms Xa and $a^T x$ looks intractable. However, we next prove that the separation problem for $\mathcal{H}$ can be solved in polynomial time to ε tolerance, which guarantees that r^+(TTRS) can be computed to ε tolerance in polynomial time. We call a point (x, X) *ε-feasible for $\mathcal{H}$* if it satisfies each constraint defining $\mathcal{H}$ with right-hand side relaxed to $1 - a^T x + \varepsilon$.

Proposition 3. *The separation problem for $\mathcal{H}$ is solvable to ε tolerance in time polynomial in n and $\log(1/\varepsilon)$.*

Proof. Let $(\bar{x}, \bar{X})$ be given. We need to determine in polynomial time whether $(\bar{x}, \bar{X})$ is ε-feasible for $\mathcal{H}$ and, if not, find $a \in \text{bd}(B)$ such that the corresponding SOC-RLT constraint is violated. Consider the following optimization over $a \in \text{bd}(B)$:

$$\min \left\{ (1 - a^T \bar{x})^2 - \|H^{1/2}(\bar{x} - \bar{X}a - (1 - a^T \bar{x})h)\|^2 : a \in \text{bd}(B) \right\}.$$

This is ε-solvable in polynomial time because it is equivalent to the classical (equality constrained) TRS, which is polynomial in n and $\log(1/\varepsilon)$. In addition, if the objective value of the approximate solution a^* is greater than or equal to 0, then we have verified that $(\bar{x}, \bar{X})$ is ε-feasible for $\mathcal{H}$, while if the optimal value is less than 0, then a^* corresponds to a violated SOC-RLT constraint. □

Corollary 3. r^+(TTRS) *can be computed to ε tolerance in time polynomial in n and* $\log(1/\varepsilon)$.

Proof. Since the separation problem for $\mathcal{H}$ is polynomial-time solvable to tolerance ε, the separation problem for the feasible set of the optimization problem defining r^+(TTRS) is also polynomial-time to tolerance ε. The corollary thus follows by the equivalence of separation and optimization via the ellipsoid method [9]. □

In the following subsections, we investigate the strength of r^+(TTRS) computationally.

5.1 Instances from Yuan and Ye-Zhang

Yuan [22] presents an instance of TTRS with $n = 2$,

$$Q = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \; c = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \; H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; h = \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

and, in particular, the radius of the ball is 2 (not 1 as above). Adjusting for the different radius, we calculate the optimal solution of the basic SDP relaxation to be

$$\bar{x} = \begin{pmatrix} 1.75 \\ 0 \end{pmatrix}, \; \bar{X} = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$$

with value $Q \bullet \bar{X} + c^T \bar{x} = -0.5$. In addition, $\text{rank}(Y(\bar{x}, \bar{X})) = 2$, providing evidence that the basic SDP relaxation is not tight. So we calculate $r^+(\text{TTRS})$ by separating the SOC-RLT cuts in $\mathcal{H}$. In fact, the single SOC-RLT cut based on the supporting hyperplane $a^T x \leq 2$, where $a = (1, 0)^T$ suffices to deliver the optimal solution

$$x^* = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \; X^* = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix},$$

which has value $Q \bullet X^* + c^T x^* = 0$ and $\text{rank}(Y(x^*, X^*)) = 1$, proving that x^* is optimal for this TTRS instance. Yuan also presents a second instance for which the basic SDP relaxation delivers an optimal, rank-1 solution $Y(x, X)$ and hence is tight.

Ye and Zhang [21] also present three instances of TTRS for which the basic SDP relaxation is not tight, i.e., $r(\text{TTRS}) < v(\text{TTRS})$. By applying the above separation routine to calculate $r^+(\text{TTRS})$, we obtain $r^+(\text{TTRS}) = v(\text{TTRS})$ for each of the three instances, and in each case, the strengthened relaxation yields a global optimal solution via a rank-1 matrix solution $Y(x, X)$.

5.2 An example with a positive gap

Despite the examples of the preceding subsection, it does not always hold that $r^+(\text{TTRS})$ equals $v(\text{TTRS})$. Consider $n = 2$ and

$$\min\{x^T Q x + c^T x : \|x\| \leq 1, \|H^{1/2} x\| \leq 1\},$$

which is a special case with concentric trust regions. The paper [21] shows that the SDP

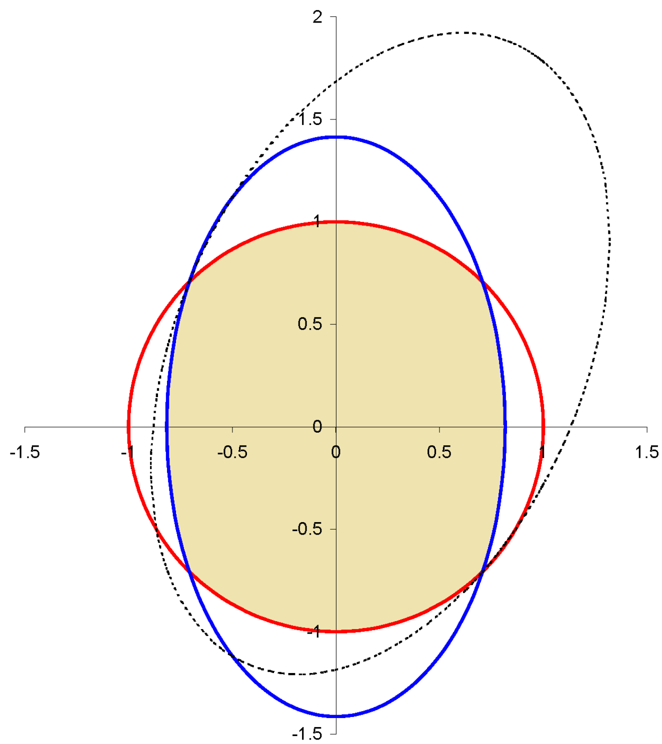


Figure 1: Counterexample with $r^+(\text{TTRS}) < v(\text{TTRS})$

relaxation is tight for such a problem if $c = 0$. On the other hand, with

$$H = \frac{1}{2}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} -4 & 1 \\ 1 & -2 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

one can verify that $v(\text{TTRS}) = -4$ at $x^* = (\pm 1, \mp 1)^T/\sqrt{2}$, while $r(\text{TTRS}) = -4.25$. In Figure 1 we illustrate the feasible region for this problem, as well as the optimal objective contour (noting that the objective is concave). The fact that $v(\text{TTRS}) = -4$ can be verified in several ways; one approach uses the result of Section 3 and the fact that the feasible region of the problem can be written as the union of the feasible regions for two instances of the form TRS1. Applying our strengthened approach yields $r^+(\text{TTRS}) \approx -4.0360$, which still leaves a 0.9% gap to the true solution value.

5.3 Random instances from Martínez

In this subsection, we explore additional TTRS instances where $r^+(\text{TTRS})$ often equals $v(\text{TTRS})$ but sometimes not. In our opinion, these experiments illustrate both the computational value of our approach and the extent to which more work is needed to solve TTRS fully.

Martínez [10] proved that the single TRS admits at most one local-nonglobal minimizer and also provided the following guarantee of when a local-nonglobal minimizer exists:

Theorem 3 ([10] Lemma 3.4)**.** *Consider the trust-region subproblem* $\min\{x^T Qx + c^T x : \|x\| \le \Delta\}$*, where the radius* Δ *is a parameter. Let* $Q = V \Lambda V^T$ *be the spectral decomposition of* Q *with ordered eigenvalues* $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ *along the diagonal matrix* Λ*. If* $\lambda_1 < \lambda_2$ *and* $[Q^T c]_1 \ne 0$*, then there exists positive* Δ_0 *such that, for all* $\Delta > \Delta_0$*, the trust-region subproblem admits both a unique global minimizer and a unique local-nonglobal minimizer.*

This theorem is potentially valuable for constructing interesting instances of TTRS. In particular, given an instance of TRS with global minimizer x^* and local-nonglobal minimizer $\bar{x}$, one can enforce another ellipsoid that cuts off x^* but leaves $\bar{x}$ feasible. For the resulting instance of TTRS, $\bar{x}$ becomes a natural candidate for the new optimal solution, although points near x^* that remain feasible are good candidates as well. Accordingly, with several candidates for the new optimal solution, one may expect this TTRS instance to be a challenging one.[1]

Based on this intuition, we generate random instances of TTRS as follows:

1. Fix the dimension n and set $\Delta = n$. We set Δ to this relatively large value since Theorem 3 indicates that larger radii tend to lead to the existence of a local-nonglobal minimizers.

2. Generate Q as a diagonal matrix with diagonal entries distributed uniformly in $[-1, 1]$, and generate c with entries also distributed uniformly in $[-1, 1]$. With a high likelihood, Q and c satisfy the assumptions of Theorem 3.

3. Solve TRS with (Q, c, Δ), and save its global optimal solution x^*. Construct an orthogonal matrix V such that $x^* = V^T e_1$. Then the TRS instance with $(\bar{Q}, \bar{c}, \bar{\Delta}) := (VQV^T, Vc, \Delta)$ has optimal solution Δe_1. Update $(Q, c, \Delta) \leftarrow (\bar{Q}, \bar{c}, \bar{\Delta})$, which is done simply to facilitate the construction of H in the next step.

4. Form an instance of TTRS by enforcing the additional ellipsoid constraint $\|H^{1/2}(x - h)\| \le \Delta$, where $h = 0$ and H is a diagonal matrix with $H_{11} = 2$ and entries $H_{22}, \ldots, H_{nn}$ generated uniformly in $[0.5, 2]$. This construction guarantees that the optimal solution of TRS from step 3, Δe_1, is *not* feasible for TTRS.

[1] We thank Henry Wolkowicz for this valuable suggestion.

For varying choices of n, we generate 1000 such TTRS instances and solve the SOCP/SDP relaxation by separating SOC-RLT cuts. We continue resolving and separating cuts until either we can find no additional cuts—indicating that we have calculated $r^+(\text{TTRS})$—or until we have separated 25 cuts. We impose this limit of 25 only because we found that some instances would keep generating non-productive cuts due to numerical issues in the solver. All computations were obtained with SeDuMi 1.3 under Matlab 7.14 (R2012a) on an Intel Core 2 Quad CPU running at 2.4 GHz with 4 KB cache and 4 GB RAM under the Ubuntu 10.04 operating system (32-bit).

Our primary interest is the gap $v(\text{TTRS}) - r^+(\text{TTRS})$, but we actually do not know $v(\text{TTRS})$. It could be calculated exactly by the trajectory-following procedure of Ye and Zhang [21], but here we adopt a simpler approach to only estimate the gap via an upper bound on $v(\text{TTRS})$. Specifically, let (x, X) be any solution satisfying the constraints of the standard SDP relaxation of TTRS. Then x is feasible for TTRS since $I \succ 0$, $H \succ 0$, and $X \succeq xx^T$ imply

$$\begin{aligned}
&\|x\|^2 = I \bullet xx^T \le I \bullet X = \text{trace}(X) \le 1 \\
&\|H^{1/2}(x-h)\|^2 = H \bullet xx^T - 2h^T Hx + h^T Hh \le H \bullet X - 2h^T Hx + h^T Hh \le 1.
\end{aligned}$$

So $x^T Qx + c^T x$ is a valid upper bound on $v(\text{TTRS})$ that could be used to estimate the gap. In our case, we choose the upper bound $\overline{v}(\text{TTRS}) := (x^*)^T Qx^* + c^T x^*$, where (x^*, X^*) is the calculated optimal solution of our relaxation, i.e., $r^+(\text{TTRS}) = Q \bullet X^* + c^T x^*$. Hence, our gap estimate is $r^+(\text{TTRS}) - \overline{v}(\text{TTRS})$. In fact, to standardize the scale, we investigate the relative gap

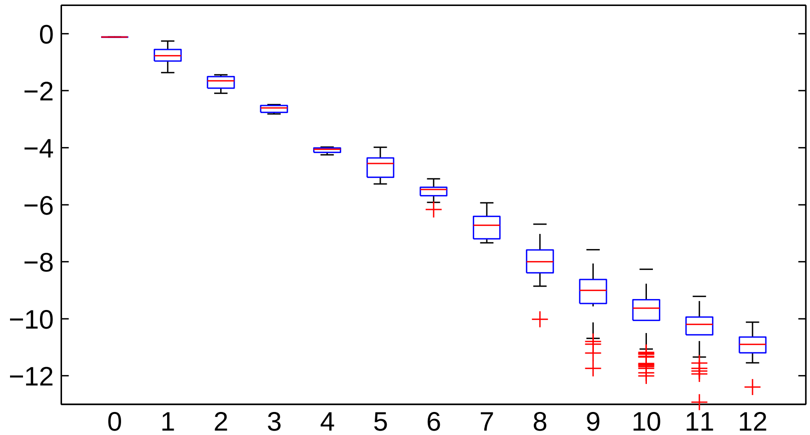
$$\frac{r^+(\text{TTRS}) - \overline{v}(\text{TTRS})}{|\overline{v}(\text{TTRS})|}.$$

We are also interested in two additional measures for each instance. First, we observe a certain "rank measure" of the optimal $Y(x^*, X^*)$ for the relaxation, which is defined as

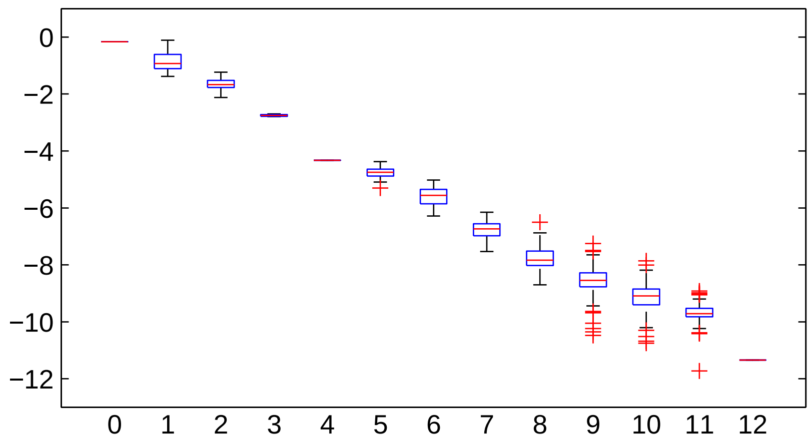
$$\frac{\lambda_n[Y(x^*, X^*)]}{\lambda_{n-1}[Y(x^*, X^*)]},$$

where λ_n and λ_{n-1} are the largest and second-largest eigenvalues, respectively. If this measure is large, then $Y(x^*, X^*)$ is numerically rank 1, but smaller values indicate rank 2 or higher. Second, we track the number of SOC-RLT cuts required to calculate $r^+(\text{TTRS})$. As it turns out, the relative gap and the number of cuts are both highly negatively correlated with the rank measure. We detail these relationships in Figures 2 and 3.

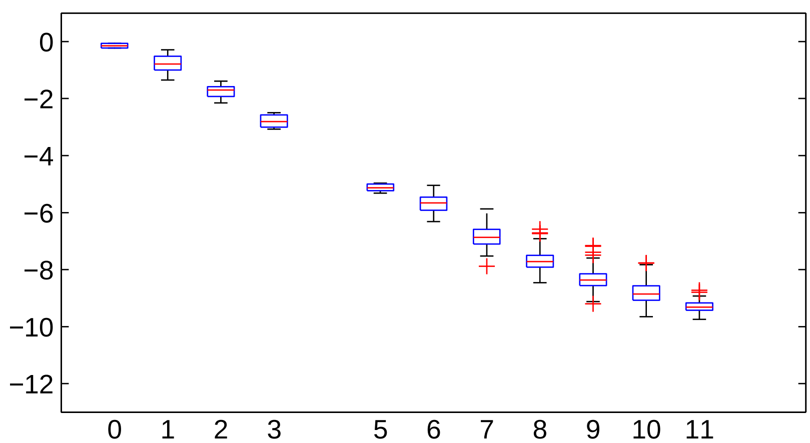
Figure 2(a) plots the relative gap (vertical axis, $\log_{10}$ scale) against the rank measure



(a) 1000 random instances with $n = 5$



(b) 1000 random instances with $n = 10$



(c) 1000 random instances with $n = 20$

Figure 2: TTRS results based on Martínez [10]. Each chart depicts the relative gap (vertical axis, $\log_{10}$ scale) versus the rank measure (horizontal axis, $\log_{10}$ scale). For each chart, the rank measure has been grouped into bins of width 1 for the box plots. A higher rank measure indicates lower numerical rank.

(horizontal axis, $\log_{10}$ scale) for 1000 random instances with $n = 5$. The instances are grouped into bins of width 1 with centers 10^0, 10^1, 10^2, etc, and a box plot is shown for each bin. It is evident that larger rank measures correlate with smaller relative gaps. In fact, the relationship appears nearly linear in the log-log scale. So the rank measure is a reliable, secondary measure of global optimality. Figures 2(b) and 2(c) show similar trends for $n = 10$ and $n = 20$.

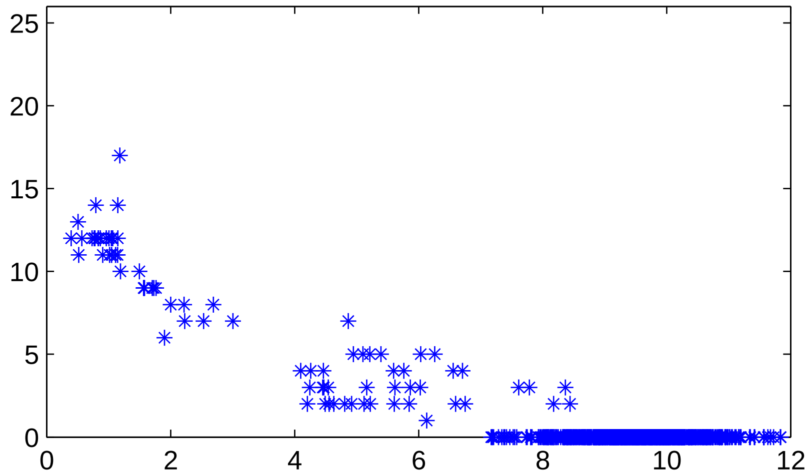
Figure 3(a) presents a scatter plot of the same 1000 instances for $n = 5$, which depicts the number of SOC-RLT cuts required (vertical axis) versus the rank measure (horizontal axis, $\log_{10}$ scale). We first note that there is a very clear inverse relationship between the number of cuts required and the rank measure, indicating that our relaxation is more likely to deliver a lower-rank solution when it requires fewer cuts. In particular, we note three groups of points in the figure. First, 92.2% of instances required 0 cuts and the corresponding rank measure was 10^7 or higher. The second group (4.0% of instances) mostly requires 1-5 cuts and achieves rank measures mostly above 10^4. The third group (3.8% of instances) mostly requires 6-15 cuts and achieves rank measures in the range 1 to 10^4. These three groups show clear jumps in the rank measure as the number of cuts changes.

In Figure 3(b), we present a similar chart for 1000 instances with $n = 10$. Here again, there is a very clear inverse trend and several distinct groups following similar patterns: 24.6% of instances require 0 cuts with high rank measure; 68.4% require about 1-5 cuts with medium rank measure; and 6.0% require about 6-20 cuts with low rank measure. There is also 1.0% that require the maximum of 25 cuts, in which case even our relaxation is not solved to optimality. Compared to $n = 5$, these percentages indicate that the $n = 10$ instances of TTRS are generally harder to solve.

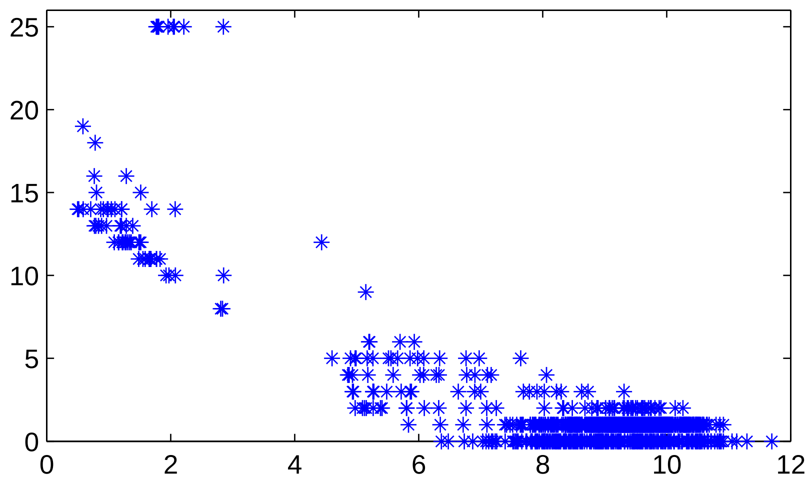
Figure 3(c) shows similar trends for $n = 20$. Using the same groupings as for $n = 10$, the percentages are 4.1%, 85.5%, 7.6%, and 2.8%, respectively. So the $n = 20$ instances are generally harder to solve than when $n = 10$.

We further summarize our experiments in Table 1, where we say that an instance is "solved by adding SOC-RLT cuts" if the instance is *not* solved by the basic SDP and the subsequent relative gap based on $r^+(\text{TTRS})$ is less than 10^{-4}. The table makes clear that the use of SOC-RLT cuts provides a substantial strengthening of the basic SDP relaxation, especially for larger dimensions.

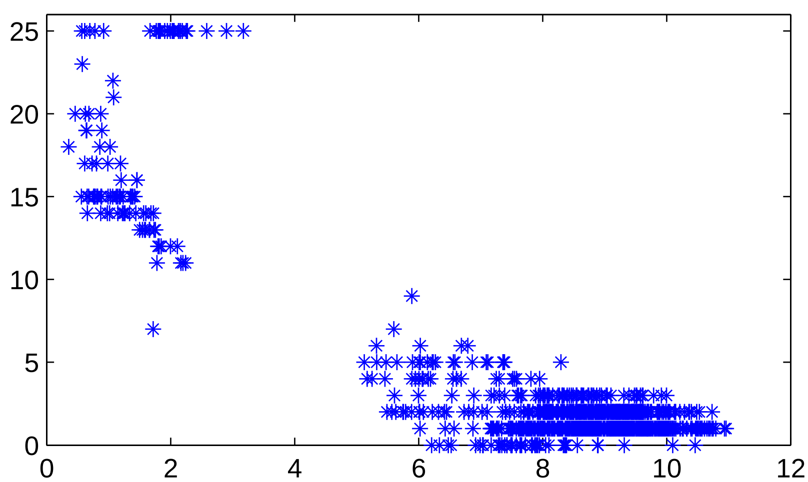
Lastly, we mention the average computational times for the 1000 instances: $n = 5$ averaged 2.5 seconds, $n = 10$ averaged 5.9 seconds, and $n = 20$ averaged 14.3 seconds.



(a) 1000 random instances with $n = 5$



(b) 1000 random instances with $n = 10$



(c) 1000 random instances with $n = 20$

Figure 3: TTRS results based on Martínez [10]. Each chart depicts the number of SOC-RLT cuts (vertical axis) versus the rank measure (horizontal axis, $\log_{10}$ scale). A higher rank measure indicates lower numerical rank.

n	% solved by basic SDP	% solved by adding SOC-RLT cuts	% unsolved
5	92.2	3.7	4.1
10	24.6	68.4	7.0
20	4.1	85.5	10.4

Table 1: Summary of outcomes on TTRS instances from Martínez

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Appendix

In this appendix, we verify the relations $x = (1-\lambda)z^1 + \lambda z^2$, $X_{\cdot 1} = (1-\lambda)z_1^1 z^1 + \lambda z_1^2 z^2$, and $\mathrm{diag}(W) \geq 0$, which appear in the proof of Lemma 4.

Proof: $x = (1-\lambda)z^1 + \lambda z^2$

Let $d := (u-x_1)^2 + (X_{11} - x_1^2) > 0$ denote the denominator of λ, and note that $d = u^2 - 2ux_1 + X_{11}$, $\lambda = d^{-1}(u-x_1)^2$ and $1-\lambda = d^{-1}(X_{11}-x_1^2)$. Also let $1 \le j \le n$. We have

$$\begin{aligned}(1-\lambda)z_j^1 &= d^{-1}(X_{11}-x_1^2)z_j^1\\ &= d^{-1}(X_{11}-x_1^2)\cdot\frac{u-x_1}{X_{11}-x_1^2}\left(X_{j1} - \left(\frac{ux_1 - X_{11}}{u-x_1}\right)x_j\right)\\ &= d^{-1}(u-x_1)\left(X_{j1} - \left(\frac{ux_1 - X_{11}}{u-x_1}\right)x_j\right)\\ &= d^{-1}\left((u-x_1)X_{j1} - (ux_1 - X_{11})\,x_j\right)\\ &= d^{-1}\left(uX_{j1} - x_1X_{j1} - ux_1x_j + X_{11}x_j\right)\end{aligned}$$

and

$$\begin{aligned}\lambda z_j^2 &= d^{-1}(u-x_1)^2 z_j^2\\ &= d^{-1}(u-x_1)^2\cdot\frac{1}{u-x_1}(ux_j - X_{j1})\\ &= d^{-1}(u-x_1)(ux_j - X_{j1})\\ &= d^{-1}(u^2x_j - uX_{j1} - ux_1x_j + x_1X_{j1}).\end{aligned}$$

Hence,

$$(1-\lambda)z_j^1 + \lambda z_j^2 = d^{-1}\left(u^2x_j - 2ux_1x_j + X_{11}x_j\right) = d^{-1}\left(u^2 - 2ux_1 + X_{11}\right)x_j = x_j.$$

Proof: $X_{\cdot 1} = (1-\lambda)z_1^1 z^1 + \lambda z_1^2 z^2$

From the previous paragraph, note in particular that

$$(1-\lambda)z_1^1 = d^{-1}(uX_{11} - x_1X_{11} - ux_1x_1 + X_{11}x_1) = d^{-1}(uX_{11} - ux_1^2) = d^{-1}(X_{11}-x_1^2)u = (1-\lambda)u,$$

which implies $z_1^1 = u$. Hence, it holds that

$$(1-\lambda)z_1^1 z_j^1 = (1-\lambda)uz_j^1 = d^{-1}u\left(uX_{j1} - x_1X_{j1} - ux_1x_j + X_{11}x_j\right).$$

In addition,

$$\begin{aligned}
\lambda z_j^2 z_1^2 &= d^{-1}(u^2 x_j - uX_{j1} - ux_1x_j + x_1X_{j1})z_1^2 \\
&= d^{-1}(u - x_1)(ux_j - X_{j1})z_1^2 \\
&= d^{-1}(u - x_1)(ux_j - X_{j1}) \cdot \frac{1}{u - x_1}(ux_1 - X_{11}) \\
&= d^{-1}(ux_j - X_{j1})(ux_1 - X_{11}) \\
&= d^{-1}\left(u^2x_1x_j - ux_jX_{11} - ux_1X_{j1} + X_{11}X_{j1}\right).
\end{aligned}$$

So

$$(1-\lambda)z_1^1 z_j^1 + \lambda z_1^2 z_j^2 = d^{-1}\left(u^2X_{j1} - 2ux_1X_{j1} + X_{11}X_{j1}\right) = X_{j1}.$$

Proof: $\text{diag}(W) \geq 0$

Using the preceding equations, we have

$$\begin{aligned}
(1-\lambda)(z_j^1)^2 &= (1-\lambda)^{-1}\left((1-\lambda)z_j^1\right)^2 \\
&= \frac{d}{X_{11} - x_1^2} \cdot d^{-2}\left(uX_{j1} - x_1X_{j1} - ux_1x_j + X_{11}x_j\right)^2 \\
&= d^{-1}(X_{11} - x_1^2)^{-1}\left(uX_{j1} - x_1X_{j1} - ux_1x_j + X_{11}x_j\right)^2 \\
&= d^{-1}(X_{11} - x_1^2)^{-1}\left[u^2X_{j1}^2 - 2ux_1X_{j1}^2 - 2u^2x_1x_jX_{j1} + 2ux_jX_{11}X_{j1} + x_1^2X_{j1}^2\right. \\
&\qquad \left. +2ux_1^2x_jX_{j1} - 2x_1x_jX_{11}X_{j1} + u^2x_1^2x_j^2 - 2ux_1x_j^2X_{11} + x_j^2X_{11}^2\right]
\end{aligned}$$

and

$$\begin{aligned}
\lambda(z_j^2)^2 &= \lambda^{-1}\left(\lambda z_j^2\right)^2 \\
&= \frac{d}{(u-x_1)^2} \cdot d^{-2}(u - x_1)^2(ux_j - X_{j1})^2 \\
&= d^{-1}(ux_j - X_{j1})^2 \\
&= d^{-1}(X_{11} - x_1^2)^{-1}(X_{11} - x_1^2)(ux_j - X_{j1})^2 \\
&= d^{-1}(X_{11} - x_1^2)^{-1}(X_{11} - x_1^2)(u^2x_j^2 - 2ux_jX_{j1} + X_{j1}^2) \\
&= d^{-1}(X_{11} - x_1^2)^{-1}\left(u^2x_j^2X_{11} - 2ux_jX_{11}X_{j1} + X_{11}X_{j1}^2 - u^2x_1^2x_j^2 + 2ux_1^2x_jX_{j1} - x_1^2X_{j1}^2\right).
\end{aligned}$$

Hence,

$$
\begin{aligned}
(1-\lambda)(z_j^1)^2 + \lambda(z_j^2)^2 &= d^{-1}(X_{11} - x_1^2)^{-1}\left[u^2 X_{j1}^2 - 2ux_1 X_{j1}^2 + X_{11}X_{j1}^2 - 2u^2 x_1 x_j X_{j1} + u^2 x_j^2 X_{11}\right. \\
&\qquad \left. +4ux_1^2 x_j X_{j1} - 2X_{11}x_1 x_j X_{j1} - 2ux_1 x_j^2 X_{11} + X_{11}x_j^2 X_{11}\right] \\
&= d^{-1}(X_{11} - x_1^2)^{-1}\left[dX_{j1}^2 - 2dx_1 x_j X_{j1} + dx_j^2 X_{11}\right] \\
&= (X_{11} - x_1^2)^{-1}\left(X_{j1}^2 - 2x_1 x_j X_{j1} + x_j^2 X_{11}\right).
\end{aligned}
$$

Using (4),

$$
\begin{aligned}
\det_j := \det \begin{pmatrix} 1 & x_1 & x_j \\ x_1 & X_{11} & X_{j1} \\ x_j & X_{j1} & X_{jj} \end{pmatrix} &= (X_{11} - x_1^2)(X_{jj} - x_j^2) - (X_{j1} - x_1 x_j)^2 \\
&= (X_{11} - x_1^2)X_{jj} - \left(X_{j1}^2 - 2x_1 x_j X_{j1} + x_j^2 X_{11}\right),
\end{aligned}
$$

and so

$$
(X_{11} - x_1^2)\left(X_{jj} - \left[(1-\lambda)(z_j^1)^2 + \lambda(z_j^2)^2\right]\right) = \det_j \geq 0,
$$

which implies $\mathrm{diag}(W) \geq 0$.