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Representing quadratically constrained quadratic programs as generalized copositive programs

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ABSTRACT

We show that any (nonconvex) quadratically constrained quadratic program (QCQP) can be represented as a generalized copositive program. In fact, we provide two representations: one based on the concept of completely positive (CP) matrices over second-order cones, and one based on CP matrices over the positive semidefinite cone.

1. Introduction

Consider the (nonconvex) quadratically constrained quadratic program (QCQP)

$$\min_{x\in\Re^n}\Big\{\langle x, Qx\rangle + 2\langle q, x\rangle \mid \langle x, Q^j x\rangle + 2\langle q^j, x\rangle \le \chi^j \quad (j = 1, \ldots, r)\Big\}, \tag{1}$$

where, in particular, Q and Q^j are general $n\times n$ symmetric matrices. Let F denote the feasible set of (1).

Problem (1) is quite general, encompassing many classes of difficult optimization problems, and thus is NP-hard. For example, (1) models quadratic programs (when all constraints are linear) and binary integer programs (when all constraints are linear except those of the form $x_i^2 = x_i$, which is equivalent to the condition $x_i \in \{0, 1\}$). Also, polynomial optimization problems may be reduced to (1) by introducing extra variables and constraints. For example, a cubic term $x_i x_j x_k$ can be made quadratic with an extra variable and constraint: $x_i y$, where $y = x_j x_k$. For more background on QCQPs, we refer the reader to [15,10].

One method for globally solving (1) is first to linearize all quadratics by introducing a new symmetric matrix variable $X = xx^T$, e.g., $\langle x, Qx\rangle + 2\langle q, x\rangle = \langle Q, X\rangle + 2\langle q, x\rangle$, where $\langle Q, X\rangle :=$ trace (QX). Then the feasible region is convexified via $\mathcal{C}'(F) :=$ cl conv $\{(x, X) \mid x \in F,\ X = xx^T\}$, which allows (1) to be cast as the equivalent problem of minimizing $\langle Q, X\rangle + 2\langle q, x\rangle$ over $(x, X) \in \mathcal{C}'(F)$. So, in a certain sense, solving (1) is equivalent to characterizing $\mathcal{C}'(F)$. In fact, many existing techniques for solving QCQPs can be interpreted as providing tractable relaxations of $\mathcal{C}'(F)$; see [3,2].

Characterizing $\mathcal{C}'(F)$ in a tractable manner is difficult; if it were easy, then we could solve (1) easily. For the case of nonconvex quadratic programming when all $Q^j = 0$, however, progress has been made using the dual concepts of *copositive* and *completely positive* (CP) matrices (see Section 1.1 below for definitions). In particular, Burer [7] has shown that every nonconvex quadratic program (QP) is equivalent to an explicit *copositive program*, which is a linear conic program over the convex cone of CP matrices. This approach focuses the difficulty of nonconvex QPs completely on the CP matrices. Said differently, any knowledge concerning CP matrices can be uniformly applied to help solve all nonconvex QPs. Fortunately, a fair amount is known about how to approximate the cone of CP matrices [4,14,17,5,6]. Burer's result also holds for specific types of quadratic constraints such as the binary condition $x_i^2 = x_i$.

Burer [8] and Eichfelder and Povh [13] extend the results of [7] to the case of nonconvex QPs with an additional convex cone constraint $x \in \mathcal{K}$. (In fact, $\mathcal{K}$ may be arbitrary in [13].) Here again, certain types of quadratic constraints are allowed [8, Section 2.4]. In this case, instead of CP matrices, the cone of interest is the generalized CP matrices over $\mathcal{K}$: $\mathcal{C}(\mathcal{K}) :=$ cl conv $\{xx^T \mid x \in \mathcal{K}\}$. In particular, Burer suggests that standard cones $\mathcal{K}$ such as the

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nonnegative orthant, the second-order cone, and the semidefinite cone (or direct products of these) could be of particular importance.

In this paper, we use the results of [8] to show how the nonconvex QCQP (1) – under the assumption of a bounded, nonempty feasible region – can be expressed as a linear conic program over a cone of the form $\mathcal{C}(\mathcal{K})$. In fact, we provide two approaches: one with $\mathcal{K}$ being, in essence, the Cartesian product of second-order cones and a nonnegative orthant, and the other with $\mathcal{K}$ being the Cartesian product of a positive semidefinite cone and a nonnegative orthant.

We remark that our results complement a recent procedure by Peña et al. [18], which recasts polynomial optimization problems (even with unbounded feasible sets) as linear conic programs over "higher-order" CP matrices. In particular, their approach involves a different generalization of CP matrices than the one we employ here.

1.1. Notation and terminology

We use $\Re^n$ to denote n-dimensional Euclidean space, and $\Re^n_+$ is the nonnegative orthant. $\mathcal{S}^n$ is the space of $n \times n$ symmetric matrices. $I_n \in \mathcal{S}^n$ denotes the $n \times n$ identity matrix. For $X \in \mathcal{S}^n$, we write $X \succeq 0$ or $X \in \mathcal{S}^n_+$ if X is positive semidefinite. The second-order cone in $\Re^n$ is defined as $\mathrm{SOC}(n) := \{x \in \Re^n \mid \sqrt{x_2^2 + \cdots + x_n^2} \le x_1\}$.

For $\mathcal{C}(\mathcal{K})$ defined above, we will be particularly interested in the case when $\mathcal{K}$ is a direct product of nonnegative orthants, second-order cones, and positive semidefinite cones. In this case, $\mathrm{conv}\{X \mid x \in \mathcal{K},\ X = xx^T\}$ is already closed (see [13, Corollary 1.5] and [19, Lemma 1]), and $\mathcal{C}(\mathcal{K})$ is a convex cone. The closure operation is also unnecessary for $\mathcal{C}'(F)$ when F is compact, which is the case in this paper. When $\mathcal{K} = \Re^n_+$, the closed convex cone $\mathcal{C}(\Re^n_+)$ is called the *completely positive cone*, and it consists of *completely positive matrices*. The dual cone $\mathcal{C}(\Re^n_+)^* \subseteq \mathcal{S}^n$ is called the *copositive cone* with *copositive matrices*. More generally, we call $\mathcal{C}(\mathcal{K})$ the *generalized completely positive cone over* $\mathcal{K}$ and its dual cone in $\mathcal{S}^n$ the *generalized copositive cone over* $\mathcal{K}$. Eichfelder and Jahn [11,12] study these cones under the name *set-semidefinite cones*. (A slight difference is that, in [11,12], set-semidefinite cones are defined in the space of square matrices, with no assumption of symmetry.) We also use the abbreviation CP for completely positive, and the abbreviation GCP for generalized CP.

Even if the elements of $\mathcal{K}$ are not typically represented as vectors, the cone $\mathcal{C}(\mathcal{K})$ may still be constructed by first representing $\mathcal{K}$ in vector form (if possible). For example, to construct $\mathcal{C}(\mathcal{S}^n_+)$, one could express $\mathcal{S}^n_+$ as a subset of $\Re^{n^2}$ with columns of the semidefinite matrix stacked in order; we use $\mathrm{vec}(\cdot)$ to denote this vectorization operator. From this point forward, we assume that the elements of $\mathcal{K}$ are encoded as vectors.

1.2. Key result from the literature

The main tool in our analysis will be the following theorem that provides GCP formulations for sets $\mathcal{C}'(F)$, where F has a certain form.

Theorem 1 (*[8, Theorem 3]*). *Consider a nonempty set of the form $\mathcal{L} \cap \mathcal{Q} \subseteq \Re^d$, where $\mathcal{L} := \{z \in \mathcal{K} : Az = b\}$ is a bounded, affine slice of a closed, convex cone $\mathcal{K}$, and*

$$\mathcal{Q} := \left\{z \mid \langle z, G^k z\rangle + 2\langle g^k, z\rangle = \gamma^k\ (k = 1, \ldots, \ell)\right\}$$

is the intersection of level sets of several quadratic functions, where γ^k is the maximum value of $\langle z, G^k z\rangle + 2\langle g^k, z\rangle$ over $z \in \mathcal{L}$. Then $\mathcal{C}'(\mathcal{L} \cap \mathcal{Q})$ equals

$$\left\{ (z, Z) \mid \begin{pmatrix} 1 & z^T \\ z & Z \end{pmatrix} \in \mathcal{C}(\Re_+ \times \mathcal{K}),\quad \begin{array}{l} Az = b, \qquad (AZA^T)_{ii} = b_i^2 \quad \forall\, i \\ \langle G^k, Z\rangle + 2\langle g^k, z\rangle = \gamma^k \quad (k = 1, \ldots, \ell) \end{array} \right\}.$$

In actuality, Theorem 1 is a variant of [8, Theorem 3]. For example, the original theorem does not require that $\mathcal{L}$ be bounded, but boundedness simplifies the statement and conditions of the theorem considerably. Also, the original theorem considers only one quadratic constraint ($\ell = 1$), but [8] discusses a direct extension to $\ell > 1$ that we employ here. Finally, the original theorem employs a quadratic constraint of the form $\langle z, G^k z\rangle + 2\langle g^k, z\rangle = \gamma_k$, where γ_k is the minimum (not maximum) value of the left-hand side over $z \in \mathcal{L}$. Of course, this is the same as the constraint $\langle z, -G^k z\rangle + 2\langle -g^k, z\rangle = -\gamma_k$, where $-\gamma_k$ is a maximum. We choose the "maximum" presentation because it better matches the quadratics that we will encounter in this paper.

2. Representation with second-order cones

In this section, we show how to represent (1) as a linear conic program over the GCP cone $\mathcal{C}(\Re_+ \times \mathcal{K})$, where $\mathcal{K}$ is a direct product of second-order cones. We make the assumption that F, the feasible region of (1), is nonempty and bounded and, in particular, is contained in the unit ball $\{x \mid \langle x, x\rangle \le 1\}$ (perhaps after a simple variable scaling).

We first argue that, by lifting up one dimension, we can represent F as the intersection of a unit sphere and a quadratically constrained convex region. We require the following proposition.

Proposition 1. *Choose $\lambda^j \ge 0$ $(j = 1, \ldots, r)$ arbitrarily, and define $\tilde{\mathcal{L}} \cap \tilde{\mathcal{Q}}$, where*

$$\tilde{\mathcal{L}} := \Big\{ w \in \Re^{n+1} \mid \langle w, w\rangle \le 1,\ \langle w, P^j w\rangle + 2\langle p^j, w\rangle \le \rho^j \quad (j = 1, \ldots, r) \Big\},$$

$$\tilde{\mathcal{Q}} := \left\{ w \in \Re^{n+1} \mid \langle w, w\rangle = 1 \right\},$$

and

$$P^j = \begin{pmatrix} Q^j + \lambda^j I & 0 \\ 0 & \lambda^j \end{pmatrix}, \qquad p^j = \begin{pmatrix} q^j \\ 0 \end{pmatrix}, \qquad \rho^j = \chi^j + \lambda^j.$$

Then $F = \pi(\tilde{\mathcal{L}} \cap \tilde{\mathcal{Q}})$, where $\pi : \Re^{n+1} \to \Re^n$ is projection onto the first n coordinates.

Proof. The proof is straightforward, by identifying x with $\pi(w)$ and using the assumption that $F \subseteq \{x \mid \langle x, x\rangle \le 1\}$. In particular, w_{n+1}^2 can be viewed as a slack term for the inequality $\langle x, x\rangle \le 1$. □

If all λ^j are sufficiently large such that $P^j \succeq 0$ (which we assume from now on), then $\tilde{\mathcal{L}} \cap \tilde{\mathcal{Q}}$ is the intersection of a sphere and a convex region defined by convex quadratic inequalities. Furthermore, problem (1) may be recast as the following optimization:

$$\min_{w \in \Re^{n+1}} \left\{ \langle w, Pw\rangle + 2\langle p, w\rangle \mid w \in \tilde{\mathcal{L}} \cap \tilde{\mathcal{Q}} \right\},$$

$$\text{where } P = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix},\ p = \begin{pmatrix} q \\ 0 \end{pmatrix}. \tag{2}$$

In order to apply Theorem 1, we next show that, by introducing auxiliary variables and constraints, we can represent $\tilde{\mathcal{L}} \cap \tilde{\mathcal{Q}}$ in the form $\mathcal{L} \cap \mathcal{Q}$, where $\mathcal{L}$ and $\mathcal{Q}$ satisfy the conditions of Theorem 1. We use the following proposition due to Alizadeh and Goldfarb [1].

Proposition 2. *For each $j = 1, \dots, r$, let $P^j = (R^j)^T R^j$ for some matrix $R^j \in \Re^{\operatorname{rank}(P^j)\times(n+1)}$. Then a point $w \in \mathbb{R}^{n+1}$ satisfies $\langle w, P^j w\rangle + 2\langle p^j, w\rangle \le \rho^j$ if and only if*

$$\begin{pmatrix} \frac{1}{2}(1+\rho^j) - \langle p^j, w\rangle \\ \frac{1}{2}(1-\rho^j) + \langle p^j, w\rangle \\ R^j w \end{pmatrix} \in \operatorname{SOC}(\operatorname{rank}(P^j)+2).$$

Now, define $d := (n+2) + \sum_{j=1}^r \left(\operatorname{rank}(P^j)+2\right)$ and introduce a long vector of variables $z = (w_0; w; w^1; \dots; w^r) \in \Re^d$, where $w_0 \in \Re$, $w \in \Re^{n+1}$, and $w^j \in \Re^{\operatorname{rank}(P^j)+2}$. Also define the following subsets of $\Re^d$: $\mathcal{Q} := \{z \mid \langle w, w\rangle = 1\}$, and

$$\mathcal{L} := \left\{ z \in \mathcal{K} \mid w_0 = 1,\ w^j = \begin{pmatrix} \frac{1}{2}(1+\rho^j) - \langle p^j, w\rangle \\ \frac{1}{2}(1-\rho^j) + \langle p^j, w\rangle \\ R^j w \end{pmatrix} (j = 1, \dots, r) \right\},$$

where

$$\mathcal{K} := \operatorname{SOC}(n+2) \times \operatorname{SOC}(\operatorname{rank}(P^1)+2) \times \cdots \times \operatorname{SOC}(\operatorname{rank}(P^r)+2).$$

Letting $\tilde{\pi} : \Re^d \longrightarrow \Re^{n+1}$ denote projection onto coordinates 2 through $n+2$, i.e., $\tilde{\pi}(z) = w$, then it is easy to see by Proposition 2 that $\tilde{\mathcal{L}} \cap \tilde{\mathcal{Q}} = \tilde{\pi}(\mathcal{L} \cap \mathcal{Q})$.

Note that we may define (A, b) to write $\mathcal{L} = \{z \in \mathcal{K} \mid Az = b\}$. Take $A = [e_0^T; A^1; \dots; A^r]$, where $e_0 \in \Re^d$ is zero except for a 1 in its first position and A^j is an appropriately defined $\Re^{\operatorname{rank}(P^j)+2} \times d$ matrix. Specifically, let all entries of A^j be zero except for the following submatrices: (i) $[(p^j)^T; -(p^j)^T; -R^j]$ at columns corresponding to w; and (ii) an identity matrix $I_{\Re^{\operatorname{rank}(P^j)+2}}$ at columns corresponding to w^j. Also take $b = [1; b^1; \dots; b^r]$ with $b^j \in \Re^{\operatorname{rank}(P^j)+2}$ as follows: all entries are 0 except for the first two, which are $\frac{1}{2}(1+\rho^j)$ and $\frac{1}{2}(1-\rho^j)$, respectively.

We also have that $\mathcal{L}$ is bounded because $\langle w, w\rangle \le w_0 = 1$, and each w^j depends affinely on w. Then, since $F \ne \emptyset$ ensures that $1 = \max_{z\in\mathcal{L}} \langle w, w\rangle$, $\mathcal{L}$ and $\mathcal{Q}$ satisfy all conditions of Theorem 1. We thus obtain a generalized CP representation of $\mathcal{C}'(\mathcal{L}\cap\mathcal{Q})$. Now, properly defining $H \in \mathcal{S}^d$ and $h \in \Re^d$, (2) can be recast as

$$\min_{z\in\Re^d} \{\langle z, Hz\rangle + 2\langle h, z\rangle \mid z \in \mathcal{L} \cap \mathcal{Q}\}.$$

Thus, together with the linearization procedure described in the Introduction, we obtain a generalized CP formulation of (1).

Corollary 1. *Problem (1) is equivalent to a linear conic program over the GCP cone $\mathcal{C}(\Re_+ \times \mathcal{K})$, where $\mathcal{K}$ is the product of $r+1$ second-order cones. The dimension of $\mathcal{K}$ is $n + 2(r+1) + \sum_{j=1}^r \operatorname{rank}(P^j)$, and the GCP formulation has $4(r+1) + 2\sum_{j=1}^r \operatorname{rank}(P^j)$ equality constraints.*

Proof. The first statement follows from the preceding discussion and Theorem 1, and the size of $\mathcal{K}$ is d, as discussed above.

Let m and m' be the number of equations in $\mathcal{L}$ and $\mathcal{Q}$, respectively. We see that $m = 1 + \sum_{j=1}^r (\operatorname{rank}(P^j)+2) = 2r + 1 + \sum_{j=1}^r \operatorname{rank}(P^j)$ and $m' = 1$. Since the representation in Theorem 1 requires $2m + m' + 1$ equations (including one that sets the top-left matrix entry to 1), the number of equations in the GCP formulation is $4(r+1) + 2\sum_{j=1}^r \operatorname{rank}(P^j)$. □

3. Representation with a semidefinite cone

In this section, we describe another representation of (1) as a generalized CP program. We assume again that $F \subseteq \{x \mid \langle x, x\rangle \le 1\}$.

First, note that (1) can be reformulated as follows:

$$\min_{x,X} \left\{ \langle Q, X\rangle + 2\langle q, x\rangle \mid \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0, \quad \langle Q^j, X\rangle + \langle q^j, x\rangle \le \chi^j \quad (j = 1, \dots, r) \quad \operatorname{trace}(X) \le 1, \quad x_i^2 - X_{ii} = 0 \quad (i = 1, \dots, n) \right\}. \tag{3}$$

The reformulation is valid because $X - xx^T \succeq 0$ with zero diagonal ensures that $X = xx^T$. Also, trace $(X) \le 1$ is equivalent to $\langle x, x\rangle \le 1$. Next, we introduce a slack vector $s \in \Re_+^{r+1}$ indexed from 0, and define the sets $\mathcal{L}$ and $\mathcal{Q}$ as

$$\mathcal{L} := \left\{ (x, X, s) \mid \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0, \quad s \ge 0, \quad \operatorname{trace}(X) + s_0 = 1, \quad \langle Q^j, X\rangle + 2\langle q^j, x\rangle + s_j = \chi^j \quad (j = 1, \dots, r) \right\},$$

$$\mathcal{Q} := \left\{ (x, X, s) \mid x_i^2 - X_{ii} = 0 \quad (i = 1, \dots, n) \right\}.$$

Then (3) is equivalent to optimizing $\langle Q, X\rangle + 2\langle q, x\rangle$ over $\mathcal{L} \cap \mathcal{Q}$.

Note that $\mathcal{L}$ can be written as $\{z \in \mathcal{K} \mid Az = b\}$ with $\mathcal{K} = \mathcal{S}_+^{n+1} \times \Re_+^{r+1}$ for a properly defined system $Az = b$. Recalling our assumption that elements of $\mathcal{K}$ are encoded as vectors – and here we assume that the matrix columns are stacked in order – one row of (A, b) fixes the "top-left" element of $(1, x^T; x, X)$ to 1; one row enforces the constraint trace $(X) + s_0 = 1$; and r rows constrain $\langle Q^j, X\rangle + 2\langle q^j, x\rangle + s_j = \chi^j$ for each j. An additional $\binom{n+1}{2}$ rows are also needed to enforce the symmetry of $(1, x^T; x, X)$. In total, $\mathcal{L}$ has $\binom{n+1}{2} + r + 2$ equations and $(n+1)^2 + r + 1$ variables. We also note that $\mathcal{Q}$ has n quadratic equations.

It is easy to see that $\mathcal{L}$ is bounded. Furthermore, because the inequalities $x_i^2 - X_{ii} \le 0$ are implied by semidefiniteness in $\mathcal{L}$, and $F \ne \emptyset$, we see that $0 = \max_{(x,X,s)\in\mathcal{L}} \left(x_i^2 - X_{ii}\right)$. So $\mathcal{L}$ and $\mathcal{Q}$ satisfy all the conditions of Theorem 1. Hence, an application of Theorem 1 and the linearization procedure described in the Introduction yields a generalized copositive formulation for (1).

Corollary 2. *Problem (1) is equivalent to a linear conic program over the GCP $\mathcal{C}(\Re_+ \times \mathcal{K})$, where $\mathcal{K} = \mathcal{S}_+^{n+1} \times \Re_+^{r+1}$. The dimension of $\mathcal{K}$ is $(n+1)^2 + r + 1$, and the GCP formulation has $(n+1)^2 + 2(r+2)$ equality constraints.*

Proof. The first statement follows from the preceding discussion and Theorem 1, and the size of $\mathcal{K}$ follows easily from the assumption that matrices are encoded as vectors by stacking columns in order.

Let $m = \binom{n+1}{2} + r + 2$ and $m' = n$ be the numbers of equations in $\mathcal{L}$ and $\mathcal{Q}$, respectively. Since the representation in Theorem 1 requires $2m + m' + 1$ equations (including one that sets the top-left matrix entry to 1), the number of equations in the GCP formulation is as claimed. □

4. Additional remarks

As mentioned in the Introduction, a fair amount is known about approximating the cone $\mathcal{C}(\Re_+^n)$ of CP matrices. While one can, in principle, approximate $\mathcal{C}(\Re_+^n)$ to any accuracy if one is willing to spend the computational effort, the following relaxation is often used in practice: $\mathcal{D}(\Re_+^n) := \{X \succeq 0 \mid X \ge 0\}$. This is the so-called set of *doubly nonnegative matrices*, and it is known that $\mathcal{C}(\Re_+^n) \subseteq \mathcal{D}(\Re_+^n)$ with equality if and only if $n \le 4$ [16].

Relatively little is known about approximating the GCP cones $\mathcal{C}(\mathcal{K})$ or $\mathcal{C}(\Re_+ \times \mathcal{K})$ as studied in this paper, even in small dimensions. In analogy with approximation hierarchies for the CP matrices, Zuluaga et al. [20] introduce generalized hierarchies that could be applied in this case.

As an alternative to their approach, however, we end the paper by proposing a direct generalization of $\mathcal{D}(\Re^n_+)$ that could potentially be of computational interest. Let $\mathcal{K}$ be a closed, convex cone, and let $\mathcal{C}(\mathcal{K})$ be the GCP cone over $\mathcal{K}$. We propose that

$$\mathcal{D}(\mathcal{K}) := \left\{X \succeq 0 \mid Xs \in \mathcal{K},\ \forall\, s \in \operatorname{Ext}(\mathcal{K}^*)\right\},$$

where $\operatorname{Ext}(\mathcal{K}^*)$ is the set of extreme rays of the dual cone $\mathcal{K}^*$ of $\mathcal{K}$. Note that, in our cases of interest, $\mathcal{K}$ is self-dual, i.e., $\mathcal{K}^* = \mathcal{K}$. When $\mathcal{K} = \Re^n_+$, this reads $\mathcal{D}(\Re^n_+) = \{X \succeq 0 \mid X_{\cdot i} \in \Re^n_+ \ (i = 1, \ldots, n)\}$, which matches the doubly nonnegative matrices. We also have the following straightforward proposition.

Proposition 3. $\mathcal{C}(\mathcal{K}) \subseteq \mathcal{D}(\mathcal{K})$.

Proof. It suffices to show that the extreme rays xx^T of $\mathcal{C}(\mathcal{K})$ are in $\mathcal{D}(\mathcal{K})$, where $x \in \mathcal{K}$. Let $s \in \operatorname{Ext}(\mathcal{K}^*)$ be arbitrary. Then $(xx^T)s = \langle x, s\rangle x \in \mathcal{K}$, as desired, because $\langle x, s\rangle \geq 0$. □

In a related, yet different, context, Burer and Anstreicher [9] have proposed something very similar to $\mathcal{D}(\mathcal{K})$, when $\mathcal{K}$ is the product of two second-order cones. Borrowing some ideas from their approach, we have been able to show that, when $\mathcal{K}$ is the direct product of nonnegative orthants and second-order cones, $\mathcal{D}(\mathcal{K})$ is tractable despite its semi-infinite presentation. In addition, when $\mathcal{K} = \Re^{n_1}_+ \times \mathrm{SOC}(n_2)$, we have $\mathcal{C}(\mathcal{K}) = \mathcal{D}(\mathcal{K})$ if and only if $n_1 = 1$ or $n_1 = n_2 = 2$. A similar result is that, when $\mathcal{K} = \mathrm{SOC}(n_1) \times \mathrm{SOC}(n_2)$ with $n_1 \leq n_2$, then $\mathcal{C}(\mathcal{K}) = \mathcal{D}(\mathcal{K})$ if and only if the same conditions on n_1 and n_2 hold. For the sake of space, we do not prove these claims here.

When $\mathcal{K}$ involves direct products with the semidefinite cone, we do not know if $\mathcal{D}(\mathcal{K})$ is tractable or if $\mathcal{C}(\mathcal{K})$ equals $\mathcal{D}(\mathcal{K})$ in some cases.

Overall, we believe that the results in this paper motivate further study of the GCP cones $\mathcal{C}(\mathcal{K})$, where $\mathcal{K}$ is the direct product of nonnegative orthants, second-order cones, and semidefinite cones.

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