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Unbounded convex sets for non-convex mixed-integer quadratic programming

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Abstract This paper introduces a fundamental family of unbounded convex sets that arises in the context of non-convex mixed-integer quadratic programming. It is shown that any mixed-integer quadratic program with linear constraints can be reduced to the minimisation of a linear function over a face of a set in the family. Some fundamental properties of the convex sets are derived, along with connections to some other well-studied convex sets. Several classes of valid and facet-inducing inequalities are also derived.

Keywords Mixed-integer non-linear programming · Global optimisation · Polyhedral combinatorics · Convex analysis

Mathematics Subject Classification 90C11 · 90C26 · 90C57

1 Introduction

A Mixed-integer quadratic program (MIQP) is an optimisation problem that can be written in the following form:

$$\min \left\{ c^T x + x^T Q x : \; Ax = b, \; x \in \mathbb{Z}_+^{n_1} \times \mathbb{R}_+^{n_2} \right\}, \tag{1}$$

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where $n = n_1 + n_2$, $c \in \mathbb{Q}^n$, $Q \in \mathbb{Q}^{n\times n}$, $A \in \mathbb{Q}^{m\times n}$, $b \in \mathbb{Q}^m$, and Q is symmetric without loss of generality.

Mixed-integer quadratic programs are a generalisation of *mixed-integer linear programs* and therefore $\mathcal{NP}$-hard to solve. On the other hand, they can be regarded as a special kind of mixed-integer non-linear program (MINLP). If Q is positive semidefinite (psd), then the objective function is convex, and one can use any method for convex MINLPs (such as those described in [4,14]). Otherwise, the objective function is non-convex, and even solving the continuous relaxation of the MIQP is an $\mathcal{NP}$-hard global optimisation problem (see, e.g., [36,40]).

Following a standard approach in global optimisation (e.g., [26]), we re-write the MIQP problem in the following form:

$$\begin{aligned} \min \quad & c^T x + q^T y \\ \text{s.t.} \quad & Ax = b \\ & y_{ij} = x_i x_j \quad (1 \le i \le j \le n) \\ & x \in \mathbb{Z}_+^{n_1} \times \mathbb{R}_+^{n_2} \\ & y \in \mathbb{R}^{\binom{n+1}{2}}, \end{aligned}$$

where $q \in \mathbb{Q}^{\binom{n+1}{2}}$ is a suitable vector representing Q. This makes the objective function linear, though at the cost of having non-linear (and non-convex) constraints linking the x and y variables.

We are interested in the convex hull of feasible pairs (x, y) to this transformed problem. This is because valid linear (or, more generally, convex) inequalities for this convex hull could be used within lower-bounding procedures or exact algorithms, based on linear (or convex) programming, for non-convex MIQPs.

In this paper, we focus on the convex sets associated with *unconstrained* non-convex MIQPs, in which the linear system $Ax = b$ is absent. Although this is a genuine limitation, we will show (in Sect. 3.6) that the convex set associated with a constrained instance is always a *face* of one of the convex sets that we study. This suggests that the valid inequalities that we derive for the unconstrained case are likely to be useful also for constrained problems. (Moreover, our inequalities are also valid for problems with *quadratic constraints*.)

It turns out that there are two serious complications. First, the convex hulls turn out not to be *closed*. Second, the closure of the convex hull turns out to be *non-polyhedral*, even when $n_2 = 0$. For these reasons, we have to combine traditional polyhedral theory (see [35]) with elements of convex analysis (see [17]). A similar strategy was used by us in [6] to study a continuous quadratic optimisation problem.

The paper is structured as follows. In Sect. 2, we review the relevant literature. In Sect. 3, we define our convex sets more formally, and establish several results concerning them, including a determination of their dimension, complexity, extreme points and rays, and affine symmetries. The next three sections study certain valid linear inequalities and their associated faces for the pure continuous case (Sect. 4), pure integer case (Sect. 5), and mixed case (Sect. 6), respectively. Then, in Sect. 7, we present complete linear descriptions for some small values of n_1 and n_2. Finally, in Sect. 8, we pose some questions for future research.

Remark An extended abstract of this paper appeared in the IPCO proceedings [20]. The results given in this full version are however much more extensive, and also more general, since [20] was concerned only with the pure integer case.

2 Literature review

In this section, we review the relevant literature. We cover matrix cones in Sect. 2.1, matrix variables in Sect. 2.2, the Boolean quadric polytope in Sect. 2.3, and some other related polytopes and convex sets in Sect. 2.4.

2.1 Matrix cones

We begin by recalling some results on matrices and related cones. A symmetric matrix $M \in \mathbb{R}^{n \times n}$ is psd if it can be factorised as AA^T for some real matrix A. The set of psd matrices of order n forms a convex, closed and pointed cone in $\mathbb{R}^{n \times n}$. The extreme rays of this cone correspond to the rank-1 psd matrices, i.e., those that can be written as vv^T for some $v \in \mathbb{R}^n$ (see, e.g., [16]).

A symmetric matrix $M \in \mathbb{R}^{n \times n}$ is called *completely positive* if it can be factorised as AA^T for some *non-negative* real matrix A [25]. The set of completely positive matrices of order n also forms a convex, closed and pointed cone in $\mathbb{R}^{n \times n}$, and the extreme rays of that cone correspond to the rank-1 completely positive matrices [3].

It is known that a symmetric matrix $M \in \mathbb{R}^{n \times n}$ is psd if and only if $v^T M v \geq 0$ for all vectors $v \in \mathbb{R}^n$. This provides a complete description of the psd cone in terms of linear inequalities. On the other hand, testing whether a matrix is completely positive is $\mathcal{NP}$-hard [9,28], which makes it unlikely that a complete linear description of the completely positive cone will ever be found. (Of course, the completely positive cone is contained in the intersection of the psd cone and the non-negative orthant $\mathbb{R}^{n \times n}_+$.)

2.2 Matrix variables

The idea of introducing new variables, which represent products of pairs of original variables, has been applied to many different problems, including non-convex quadratically-constrained programs [11,33,38], 0–1 linear programs [23,36] and 0–1 quadratic programs [21,31]. It is common practice to view those variables as being arranged in a symmetric matrix.

Specifically, given an arbitrary vector $x \in \mathbb{R}^n$, consider the matrix $X = xx^T$. Note that X is real, symmetric and psd, and that, for $1 \leq i \leq j \leq n$, the entry X_{ij} is nothing but our variable y_{ij}. Moreover, as pointed out in [23], the augmented matrix

$$\hat{X} := \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}$$

is also psd. This fact enables one to construct useful semidefinite programming (SDP) relaxations of various $\mathcal{NP}$-hard optimisation problems (e.g., [11,15,21,23, 31,33,38]).

Clearly, if $x \in \mathbb{R}^n_+$, then $\hat{X}$ is completely positive rather than merely psd. One can use this fact to derive stronger SDP relaxations; see the survey [10].

2.3 The Boolean quadric polytope

The *Boolean quadric polytope* is a polytope associated with unconstrained 0-1 quadratic programs. The Boolean quadric polytope of order n, which we will denote by BQP_n, is defined as:

$$\text{BQP}_n = \text{conv}\left\{(x, y) \in \{0, 1\}^{n+\binom{n}{2}} : y_{ij} = x_i x_j \ (1 \le i < j \le n)\right\}.$$

Note that here, there is no need to define the variable y_{ij} when $i = j$, since squaring a binary variable has no effect.

Padberg [30] derived various valid and facet-defining inequalities for BQP_n, called *triangle*, *cut* and *clique* inequalities. A class of inequalities that includes all of Padberg's inequalities as a special case was introduced by Boros and Hammer [5]. These take the form:

$$\sum_{i=1}^{n} v_i(v_i + 2s + 1)x_i + 2 \sum_{1 \le i < j \le n} v_i v_j y_{ij} + s(s + 1) \ge 0 \quad \left(\forall v \in \mathbb{Z}^n, s \in \mathbb{Z}\right). \tag{2}$$

We will call these simply *Boros-Hammer inequalities*. To see that they are valid, simply note that $(v^T x + s)(v^T x + s + 1) \ge 0$ when v and s are integral and x is binary. Expanding this quadratic inequality, replacing $x_i x_j$ by y_{ij} and x_i^2 by x_i where possible, yields (2).

Many other valid and facet-defining inequalities have been discovered for BQP_n. For an excellent survey, we refer the reader to the book [8].

2.4 Other related polytopes and convex sets

There are several other papers on polytopes related to quadratic versions of traditional combinatorial optimisation problems. Among them, we mention only [18] on the quadratic assignment polytope, [37] on the quadratic semi-assignment polytope, and [15] on the quadratic knapsack polytope.

There are also three papers on the following (non-polyhedral) convex set [1,6,42]:

$$\text{conv}\left\{x \in [0, 1]^n, \ y \in \mathbb{R}^{\binom{n+1}{2}}, \ y_{ij} = x_i x_j \ (1 \le i \le j \le n)\right\}.$$

This convex set is associated with non-convex quadratic programming with box constraints, a classical problem in global optimisation. As mentioned in the introduction, we used in [6] a combination of polyhedral theory and convex analysis to analyse this convex set.

Finally, we mention that Saxena et al. [34] described a lift-and-project technique for generating valid inequalities for non-convex MIQPs.

3 The convex sets and their basic properties

In this section, we define the convex sets formally and then establish some of their basic properties.

3.1 Definitions

For a given pair (n_1, n_2) of non-negative integers, let:

$$F^+_{n_1,n_2} = \left\{(x, y) \in \left(\mathbb{Z}^{n_1}_+ \times \mathbb{R}^{n_2}_+\right) \times \mathbb{R}^{\binom{n}{2}}_+ \ : \ y_{ij} = x_i x_j \ (1 \le i \le j \le n)\right\}.$$

We are interested in the convex hull of $F^+_{n_1,n_2}$. Unfortunately, we immediately face the following complication:

Proposition 1 *The convex hull of $F^+_{n_1,n_2}$ is not closed.*

Proof First, we show that $F^+_{1,0}$ is not closed. For any $t \in \mathbb{Z}_+$, let (x^t, y^t) be the member of $F^+_{1,0}$ that arises when $(x_1, y_{11}) = (t, t^2)$. Moreover, for $t > 0$, let

$$\left(\tilde{x}^t, \tilde{x}^t\right) = \frac{1}{t^2}\left(x^t, y^t\right) + \frac{t^2 - 1}{t^2}\left(x^0, y^0\right) = \left(t^{-1}, 1\right).$$

Note that $(\tilde{x}^t, \tilde{x}^t)$ is a convex combination of members of $F^+_{1,0}$ and therefore lies in the convex hull. However, $\lim_{t\to\infty}(\tilde{x}^T, \tilde{y}^T) = (0, 1)$ does not lie in conv $F^+_{1,0}$ because any $(x, y) \in$ conv $F^+_{1,0}$ with $x = 0$ must have $y = 0$. Since the convex hull does not contain all of its limit points, it is not closed.

Now suppose that $n_1 > 0$. Then $F^+_{1,0}$ is the face of $F^+_{n_1,n_2}$ induced by the valid inequalities $y_{ii} \ge 0$ for all $i > 1$. Since this face is not closed, neither is $F^+_{n_1,n_2}$ itself.

A similar argument shows that $F^+_{0,1}$ is not closed, and therefore that $F^+_{n_1,n_2}$ is not closed when $n_2 > 0$. □

We are therefore led to look at the closure of the convex hull, which we denote by $\text{MIQ}^+_{n_1,n_2}$. Figure 1 represents $\text{MIQ}^+_{1,0}$. Observe that $\text{MIQ}^+_{1,0}$, despite having facets, is not a polyhedron. (A polyhedron is defined as the intersection of a *finite* number of half-spaces, but $\text{MIQ}^+_{1,0}$ is the intersection of a *countably infinite* number of half-spaces.) Moreover, it is easy to see that $\text{MIQ}^+_{0,1}$ is a convex set with a curved boundary. Indeed, $\text{MIQ}^+_{n_1,n_2}$ is never polyhedral.

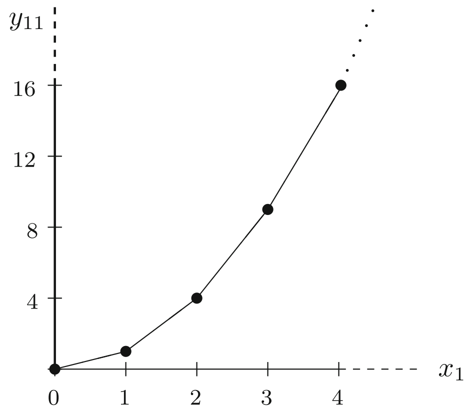

Fig. 1 The convex set $\mathrm{MIQ}^+_{1,0}$

For the purposes of what follows, we introduce a version of $F^+_{n_1,n_2}$ that does not involve non-negativity. More precisely, we define:

$$F_{n_1,n_2} := \left\{(x, y) \in \left(\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}\right) \times \mathbb{R}^{\binom{n+1}{2}} \;:\; y_{ij} = x_i x_j \;(1 \le i \le j \le n)\right\}.$$

One can show that the convex hull of F_{n_1,n_2} is closed when $n_1 = 0$ and when $(n_1, n_2) = (1, 0)$. We will show in Sect. 3.4 that it is not closed when $n_1 \ge 2$. We do not know if it is closed when $n_1 = 1$ and $n_2 > 0$. In any case, in what follows, we will work with the closure of the convex hull, which we denote by MIQ_{n_1,n_2}.

3.2 Complexity

Next, we present some complexity results.

Proposition 2 *Minimising a linear function over* $\mathrm{MIQ}^+_{0,n}$, $\mathrm{MIQ}_{n,0}$ *or* $\mathrm{MIQ}^+_{n,0}$ *is* $\mathcal{NP}$-*hard in the strong sense.*

Proof It follows from the definitions that these three problems are equivalent to minimising an arbitrary quadratic function over $\mathbb{R}^n_+$, $\mathbb{Z}^n$ or $\mathbb{Z}^n_+$, respectively. The first problem was shown to be strongly $\mathcal{NP}$-hard in [28]. The second problem includes as a special case the well-known closest vector problem (CVP), which takes the form:

$$\min\left\{\|Bx - t\|_2 \,:\, x \in \mathbb{Z}^n\right\},$$

where $B \in \mathbb{Z}^{n\times n}$ and $t \in \mathbb{Q}^n$. The CVP was shown to be strongly $\mathcal{NP}$-hard in [41]. As for the third problem, one can reduce the CVP to that as well, by writing it as:

$$\min\left\{\|B(x - x') - t\|_2 \,:\, x, x' \in \mathbb{Z}^n_+\right\}.$$

Thus, the third problem is also strongly $\mathcal{NP}$-hard. □

Proposition 3 *Minimising a linear function over* $\mathrm{MIQ}_{0,n}$ *is polynomial-time solvable.*

Proof This is equivalent to minimising an arbitrary quadratic function over $\mathbb{R}^n$. If the quadratic function is convex, the problem can be solved by elementary linear algebra. If not, the problem is unbounded. □

Proposition 2 suggests that there is no hope of obtaining complete linear descriptions of $\text{MIQ}^+_{n,0}$, $\text{MIQ}^+_{0,n}$ or $\text{MIQ}_{n,0}$ for general n. On a more positive note, we have the following result:

Proposition 4 *Minimising a linear function over* $\text{MIQ}^+_{0,n}$ *or* $\text{MIQ}_{n,0}$ *is solvable in polynomial time when n is fixed.*

Proof When n is fixed, one can minimise an arbitrary quadratic function over $\mathbb{R}^n_+$ by enumerating all of the faces of $\mathbb{R}^n_+$, and solving a Karush–Kuhn–Tucker system for each face. So consider minimising an arbitrary quadratic function over $\mathbb{Z}^n$. If the quadratic function is not convex, the problem is easily shown to be unbounded. If, on the other hand, the quadratic function is convex, then the problem can be solved for fixed n with the algorithm described in [19]. □

There is therefore some hope of obtaining complete linear descriptions of $\text{MIQ}^+_{0,n}$ and $\text{MIQ}_{n,0}$ for small values of n. We do not know the complexity of minimising a linear function over $\text{MIQ}^+_{n,0}$ for fixed n.

3.3 Dimension

We next establish the dimensions of $\text{MIQ}^+_{n_1,n_2}$ and MIQ_{n_1,n_2}.

Proposition 5 *For all* $n = n_1+n_2$, *both* $\text{MIQ}^+_{n_1,n_2}$ *and* MIQ_{n_1,n_2} *are full-dimensional, i.e., of dimension* $n + \binom{n+1}{2}$.

Proof Consider the following points in $\text{MIQ}^+_{n_1,n_2}$:

- the origin (i.e., all variables set to zero);
- for $i = 1, \ldots, n$, the point having $x_i = y_{ii} = 1$ and all other variables zero;
- for $i = 1, \ldots, n$, the point having $x_i = 2$, $y_{ii} = 4$ and all other variables zero;
- for $1 \le i < j \le n$, the point having $x_i = x_j = 1$, $y_{ii} = y_{jj} = y_{ij} = 1$, and all other variables zero.

These $n + \binom{n+1}{2} + 1$ points are easily shown to be affinely independent, and therefore $\text{MIQ}^+_{n_1,n_2}$ is full-dimensional. Since $\text{MIQ}^+_{n_1,n_2}$ is contained in MIQ_{n_1,n_2}, the same is true for MIQ_{n_1,n_2}. □

3.4 Extreme points and rays

Next, we characterise the extreme points and rays of $\text{MIQ}^+_{n_1,n_2}$ and MIQ_{n_1,n_2}.

Lemma 1 *The extreme points of* $\text{MIQ}^+_{n_1,n_2}$ *and* MIQ_{n_1,n_2} *are the members of* $F^+_{n_1,n_2}$ *and* F_{n_1,n_2}*, respectively.*

Proof From the definition of $\text{MIQ}^+_{n_1,n_2}$, each one of its extreme points must be a member of $F^+_{n_1,n_2}$. Moreover, given any vector $x^* \in \mathbb{Z}^{n_1}_+ \times \mathbb{R}^{n_2}_+$, there is a (convex) quadratic function that achieves its minimum uniquely at x^*. Accordingly, given any pair $(x^*, y^*) \in F^+_{n_1,n_2}$, there is a linear function such that the minimum of that function over $\text{MIQ}^+_{n_1,n_2}$ is achieved only at (x^*, y^*). A similar argument applies to MIQ_{n_1,n_2}. □

Theorem 1 *Consider the following two sets, which are affine images of the extreme rays of the completely positive and psd cones, respectively:*

$$G^+_{0,n} = \left\{ y \in \mathbb{R}^{\binom{n+1}{2}} \; : \; \exists \, x \in \mathbb{R}^n_+ \text{ s.t. } (x, y) \in F^+_{0,n} \right\}$$
$$G_{0,n} = \left\{ y \in \mathbb{R}^{\binom{n+1}{2}} \; : \; \exists \, x \in \mathbb{R}^n \text{ s.t. } (x, y) \in F_{0,n} \right\}.$$

The sets of extreme rays of $\text{MIQ}^+_{n_1,n_2}$ *and* MIQ_{n_1,n_2} *are* $\left\{(0, y) \; : \; y \in G^+_{0,n}\right\}$ *and* $\left\{(0, y) \; : \; y \in G_{0,n}\right\}$ *respectively.*

Proof We prove the free case; the nonnegative case is similar.

Let $(\Delta x, \Delta y)$ be a ray of MIQ_{n_1,n_2} and let ΔX be the symmetric matrix corresponding to Δy. From the result of Lovász and Schrijver [23] mentioned in Sect. 2.2, the augmented matrix

$$\begin{pmatrix} 1 & M\Delta x^T \\ M\Delta x & M\Delta X \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + M \begin{pmatrix} 0 & \Delta x^T \\ \Delta x & \Delta X \end{pmatrix}$$

must be psd for all $M \in \mathbb{R}_+$. This implies that $\Delta x = 0$. It also implies that ΔX is psd, which means that it is the sum of rank-1 psd matrices. Equivalently, Δy is the sum of members of $G_{0,n}$.

To complete the proof, we show that, for each $y^* \in G_{0,n}$, the vector $(0, y^*)$ is an extreme ray of MIQ_{n_1,n_2}. So, let x^* be the vector corresponding to y^*, and let M be an arbitrarily large positive integer. We can decompose Mx^* into an integral part and a (possibly) fractional part by writing $Mx^* = \tilde{x} + \epsilon$, where $\tilde{x} \in \mathbb{Z}^n_+$ and $\epsilon \in [0, 1)^n$. Let $(\tilde{x}, \tilde{y})$ be the member of F_{n_1,n_2} corresponding to $\tilde{x}$. We have:

$$\begin{aligned} y^*_{ij} &= M^{-2} \left(\tilde{x}_i \tilde{x}_j + \epsilon_i \epsilon_j + \tilde{x}_i \epsilon_j + \epsilon_i \tilde{x}_j \right) \\ &= M^{-2} \tilde{y}_{ij} + M^{-2} \left(\epsilon_i \epsilon_j + \tilde{x}_i \epsilon_j + \epsilon_i \tilde{x}_j \right). \end{aligned}$$

Now, since the origin is also a member of F_{n_1,n_2}, the vector $M^{-2}(\tilde{x}, \tilde{y})$ belongs to conv F_{n_1,n_2}. Moreover, as M increases, $M^{-2}(\tilde{x}, \tilde{y})$ approaches arbitrarily closely to $(0, y^*)$. Therefore, $(0, y^*)$ lies in the closure of conv F_{n_1,n_2}, and so does any positive multiple of it. It is therefore a ray of MIQ_{n_1,n_2}. Moreover, it is extreme, since the associated symmetric matrix (say, X^*) has rank 1, and every rank-1 matrix is an extreme ray of the psd cone. □

The following two results then arise as fairly simple corollaries:

Corollary 1 *The projection of* $\text{MIQ}^+_{n_1,n_2}$ *into y-space is an affine image of the completely positive cone of order n, and the projection of* MIQ_{n_1,n_2} *into y-space is an affine image of the psd cone of order n.*

Proof By Lemma 1, if (x^*, y^*) is an extreme point of MIQ_{n_1,n_2}, then the corresponding symmetric matrix X^* lies in the psd cone of order n. By Theorem 1, $(0, \Delta y)$ is a ray of MIQ_{n_1,n_2} if and only if the corresponding matrix ΔX^* lies in the psd cone of order n. For $\text{MIQ}^+_{n_1,n_2}$, just replace 'psd' with 'completely positive'. □

Corollary 2 *The convex hull of* F_{n_1,n_2} *is not closed when* $n_1 \geq 2$.

Proof By setting $x = (1, \sqrt{2}, 0, \ldots, 0)^T$ in Theorem 1, we obtain an extreme ray of MIQ_{n_1,n_2} with $y_{11} = 1$, $y_{22} = 2$, $y_{12} = \sqrt{2}$ and all other y variables equal to 0. Since $\sqrt{2}$ is irrational, this cannot be a ray of conv F_{n_1,n_2}. □

3.5 Affine symmetries

Now we examine the affine symmetries of $\text{MIQ}^+_{n_1,n_2}$ and MIQ_{n_1,n_2}, i.e., affine transformations that map the convex sets onto themselves. It turns out that these are closely related to the affine symmetries of the corresponding subsets of $\mathbb{R}^n$:

Proposition 6 *Let T be an affine transformation that maps the set* $\mathbb{Z}^{n_1}_+ \times \mathbb{R}^{n_2}_+$ *(respectively,* $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$*) onto itself. There exists an affine transformation* T' *that maps* $\text{MIQ}^+_{n_1,n_2}$ *(respectively,* MIQ_{n_1,n_2}*) onto itself, and maps any point* (x, y) *onto a point* (x', y') *with* $x' = T(x)$.

Proof Let $T(x) = Ax + b$, where $A \in \mathbb{R}^{n \times n}$ is non-singular and $b \in \mathbb{R}^n$. Given any pair (x, y), let X be the symmetric matrix associated with y as usual. Let $\tilde{T}$ be the affine mapping that maps X onto $AXA^T + (Ax)b^T + b(x^T A^T) + bb^T$. Let T' be the affine mapping that maps x onto $T(x)$, and maps y onto the vector corresponding to the matrix $\tilde{T}(X)$. Observe that, when (x, y) is an extreme point of either $\text{MIQ}^+_{n_1,n_2}$ or MIQ_{n_1,n_2}, we have $X = xx^T$ and $\tilde{T}(X) = (Ax + b)(Ax + b)^T = x'(x')^T$. Then, the point $T'(x, y) = (x', y')$ satisfies $y'_{ij} = x'_i x'_j$ for all $1 \leq i \leq j \leq n$, and is therefore also an extreme point. □

Remark 1 The only affine transformations that map $\mathbb{Z}^{n_1}_+$ onto itself are the rotations that permute the indices $1, \ldots, n_1$. The only affine transformations that map $\mathbb{R}^{n_2}_+$ onto itself are those consisting of rotations that permute the indices $1, \ldots, n_2$, together with 'stretches' that map x onto Dx, where D is a nonnegative diagonal matrix. Thus, the affine symmetries of $\text{MIQ}^+_{n_1,n_2}$ are rather uninteresting linear symmetries.

Remark 2 The affine transformations that map $\mathbb{Z}^{n_1}$ onto itself are those of the form $Ux + w$, where U is any unimodular integral square matrix of order n_1, and w is any integer vector of order n_1. The affine transformations that map $\mathbb{R}^{n_2}$ onto itself are those of the form $Ax + b$, where A is any non-singular square matrix of order n_2 and b is any vector of order n_2. Thus, the affine symmetries of MIQ_{n_1,n_2} are non-trivial.

Since there are an infinite number of unimodular integral square matrices of any order, we have:

Corollary 3 *Any facet of* $\text{MIQ}_{n,0}$ *is affinely congruent to a countably infinite number of other facets.*

Next, we note that it is possible to convert any facet-inducing inequality for $\text{MIQ}_{n,0}$ into a facet-inducing inequality for $\text{MIQ}^+_{n,0}$:

Theorem 2 *Suppose the inequality* $\alpha^T x + \beta^T y \geq \gamma$ *induces a facet of* $\text{MIQ}_{n,0}$. *Then there exists a vector* $t \in \mathbb{Z}^n_+$ *such that the inequality*

$$(\alpha - 2Bt)^T x + \beta^T y \geq \gamma + \alpha^T t - \beta^T w \tag{3}$$

induces a facet of $\text{MIQ}^+_{n,0}$, *where:*

- *B is the symmetric matrix defined by $B_{ii} = \beta_{ii}$ and $B_{ij} = \frac{1}{2}\beta_{ij}$ for $i < j$;*
- *$w_{ij} = t_i t_j$ for $i \leq j$.*

Proof Let $d = n + \binom{n+1}{2}$. Since the original inequality $\alpha^T x + \beta^T y \geq \gamma$ induces a facet of $\text{MIQ}_{n,0}$, there exist d affinely-independent members of $F_{n,0}$ that satisfy it at equality. Let $(x^1, y^1), \ldots, (x^d, y^d)$ denote these points. Using the definition of B, we have $\alpha^T x^j + (x^j)^T B x^j = \gamma$.

Now, for $i = 1, \ldots, n$, set $t_i := -\min\{0, \min_{1 \leq j \leq d} x_i^j\}$, and define the shifted points $\tilde{x}^j := x^j + t$ for all j. In particular, $t \in \mathbb{Z}^n_+$ and $\tilde{x}^j \in \mathbb{Z}^n_+$. Also, define $(\tilde{x}^j, \tilde{y}^j)$ to be the corresponding members of $F^+_{n,0}$. Then $(\tilde{x}^1, \tilde{y}^1), \ldots, (\tilde{x}^d, \tilde{y}^d)$ are d affinely independent members that satisfy

$$\alpha^T \left(\tilde{x}^j - t\right) + \left(\tilde{x}^j - t\right)^T B \left(\tilde{x}^j - t\right) = \gamma$$

or, equivalently,

$$(\alpha - 2Bt)^T \tilde{x}^j + \beta^T \tilde{y}^j = \gamma + \alpha^T t - \beta^T w.$$

It remains to show that the claimed inequality is actually valid for $\text{MIQ}^+_{n,0}$. Let $(\tilde{x}, \tilde{y})$ be any member of $F^+_{n,0}$, and define $(x, y) \in F_{n,0}$ with $x = \tilde{x} - t$. Then, by the logic of the previous paragraph, $(\alpha - 2Bt)^T \tilde{x} + \beta^T \tilde{y} \geq \gamma + \alpha^T t - \beta^T w$ if and only if $\alpha^T x + \beta^T y \geq \gamma$, which holds by assumption. □

Therefore, any inequality inducing a facet of $\text{MIQ}_{n,0}$ yields a countably infinite family of facet-inducing inequalities for $\text{MIQ}^+_{n,0}$ as well.

3.6 Connection with the constrained case

Now suppose that an MIQP instance is *constrained*, i.e., that a system $Ax = b$ of linear equations is present. Intuitively, one could reduce such an instance to an unconstrained one, by adding the quadratic penalty term $M||Ax - b||_2^2$ to the objective function, for a 'sufficiently large' positive scalar M. The determination of an appropriate value for M is, however, unclear. Fortunately, this doesn't matter, since we can instead optimise the original linear function over a *face* of $\text{MIQ}^+_{n_1,n_2}$. This is expressed in the following proposition:

Proposition 7 *Let $Ax = b$ be a system of p linear equations, and let $a^k x = b_k$ denote the kth such equation, for $k = 1, \ldots, p$. Let $E(A, b)$ denote the set of points (x, y) in $F^+_{n_1,n_2}$ that satisfy $Ax = b$, and let $C(A, b)$ denote the closure of the convex hull of $E(A, b)$. Then $C(A, b)$ is nothing but the face of* $\text{MIQ}^+_{n_1,n_2}$ *defined by the following p valid linear inequalities:*

$$\sum_{i=1}^{n} \left(a_i^k\right)^2 y_{ii} + 2 \sum_{1 \le i \le j \le n} a_i^k a_j^k y_{ij} - (2b_k)\, a^k \cdot x + b_k^2 \ge 0 \quad (k = 1, \ldots, p). \quad (4)$$

Proof The fact that the linear inequalities (4) are valid for $\text{MIQ}^+_{n_1,n_2}$ follows from the fact that all vectors $x \in \mathbb{R}^n$ satisfy the convex quadratic inequalities $(a^k \cdot x - b_k)^2 \ge 0$ for $k = 1, \ldots, p$, together with the fact that all points (x, y) in $F^+_{n_1,n_2}$ satisfy $y_{ij} = x_i x_j$ for all $1 \le i \le j \le n$. For the same reasons, $E(A, b)$ is nothing but the set of members of $F^+_{n_1,n_2}$ that satisfies the inequalities (4) at equality.

Now, let F be the face of $\text{MIQ}^+_{n_1,n_2}$ in question. Lemma 1 implies that the set of extreme points of F is $E(A, b)$. Moreover, Theorem 1 implies that the extreme rays of F, if any, are the vectors $(0, \Delta y)$ such that the symmetric matrix ΔX corresponding to Δy is equal to xx^T for some vector $x \in \mathbb{Z}^{n_1}_+ \times \mathbb{R}^{n_2}_+$ satisfying $Ax = 0$. But these vectors are extreme rays of $C(a, b)$ as well, since, if x^1 satisfies $Ax = b$ and x^2 satisfies $Ax = 0$, then $x^1 + \lambda x^2$ satisfies $Ax = b$ for any $\lambda \in \mathbb{R}_+$. □

4 The continuous case ($n_1 = 0$)

This section presents some results concerned with the (relatively) easy case in which all variables are continuous, i.e., in which $n_1 = 0$.

4.1 Conic characterisation

The following proposition gives a characterisation of $\text{MIQ}_{0,n}$ and $\text{MIQ}^+_{0,n}$ in terms of matrix cones:

Proposition 8 *Given a pair (x^*, y^*), let X^* be the symmetric matrix corresponding to y^*, and let*

$$\hat{X}^* = \begin{pmatrix} 1 & (x^*)^T \\ x^* & X^* \end{pmatrix}$$

be the corresponding augmented matrix. Then (x^*, y^*) lies in $\text{MIQ}_{0,n}$ *if and only if $\hat{X}^*$ is psd, and (x^*, y^*) lies in* $\text{MIQ}^+_{0,n}$ *if and only if $\hat{X}$ is completely positive.*

Proof Necessity was already pointed out in Sect. 2.2. We prove sufficiency. Note that, if C is a closed convex cone and H is a hyperplane passing through the interior of C, then any point in $C \cap H$ is a convex combination of extreme points of $C \cap H$, and all such extreme points are also extreme rays of C. Setting C to be the psd cone of order $n+1$ and H to be the hyperplane enforcing that the top-left entry of $\hat{X}^*$ must be equal to 1, we see that, if $\hat{X}^*$ is psd, then it can be expressed as a convex combination of rank-1 matrices of the form

$$\begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix}.$$

By Lemma 1, each such rank-1 matrix corresponds to an extreme point of $\text{MIQ}_{0,n}$. The case of $\text{MIQ}^+_{0,n}$ is similar. □

4.2 Psd inequalities

The next lemma introduces a class of valid inequalities:

Lemma 2 *For any non-zero vector $v \in \mathbb{R}^n$ and scalar $s \in \mathbb{R}$, the following 'psd' inequality is valid for both* $\text{MIQ}_{0,n}$ *and* $\text{MIQ}^+_{0,n}$*:*

$$(2s)v^T x + \sum_{i=1}^{n} v_i^2 y_{ii} + 2 \sum_{1 \le i < j \le n} v_i v_j y_{ij} + s^2 \ge 0. \tag{5}$$

Proof If a matrix M is psd, then $v^T M v \ge 0$ for all non-zero $v \in \mathbb{R}^n$. Applying this to the matrix $\hat{X}$ we find that:

$$\begin{pmatrix} s & v^T \end{pmatrix} \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \begin{pmatrix} s \\ v \end{pmatrix} \ge 0 \tag{6}$$

for all v and s. The correspondence between X and y then yields the desired inequalities. □

To our knowledge, the validity of the psd inequalities (5) for extended formulations of quadratic optimisation problems was first observed by Ramana [33]. Note that the inequalities (4) in Sect. 3.6 are psd inequalities.

It turns out that the psd inequalities yield a complete description of $\text{MIQ}_{0,n}$:

Proposition 9 *The psd inequalities provide a complete and non-redundant linear description of* $\text{MIQ}_{0,n}$*, and each such psd inequality induces a maximal face of dimension* $\binom{n+1}{2} - 1$.

Proof It is known (e.g., [16]) that the inequalities $v^T M v \ge 0$ for all non-zero $v \in \mathbb{R}^n$ provide a complete and non-redundant linear description of the cone of psd matrices of order n, and that each such inequality induces a maximal face of dimension $\binom{n}{2}$. Now, let $\mathcal{S}$ denote the set of matrices $\hat{X}$ that are psd. Since $\mathcal{S}$ is obtained by intersecting the psd cone of order $n+1$ with a hyperplane (see proof of Proposition 8), the inequalities (6) provide a complete and non-redundant linear description of $\mathcal{S}$, and each such inequality induces a maximal face of dimension $\binom{n+1}{2}-1$. The result then follows from Proposition 8 and the fact that the mapping from $\mathcal{S}$ to $\text{MIQ}_{0,n}$ is a linear mapping that preserves dimension. □

The psd inequalities are of course valid for $\text{MIQ}^+_{0,n}$ as well. Using the same proof technique as in Sect. 4 of our earlier paper [6], one can prove the following:

Proposition 10 *Let $v \in \mathbb{R}^n$ and $s \in \mathbb{R}$ be given. The psd inequality (5) induces a proper face of $\text{MIQ}^+_{0,n}$ if and only if there exists a point $x^* \in \mathbb{R}^n_+$ such that $v^T x^* + s = 0$. This face is maximal if and only if there exists such a point x^* in which all components are positive. If it is maximal, it has dimension $\binom{n+1}{2}-1$.*

4.3 Non-negativity inequalities

Since $\text{MIQ}^+_{0,n}$ is contained in the completely positive cone, it is clear that all variables are constrained to be non-negative. The following theorem states conditions under which non-negativity inequalities induce facets of $\text{MIQ}^+_{0,n}$.

Theorem 3 *The inequalities $x_i \geq 0$ for all $1 \leq i \leq n$, and the inequalities $y_{ij} \geq 0$ for all $1 \leq i < j \leq n$, induce facets of $\text{MIQ}^+_{0,n}$. The inequalities of the form $y_{ii} \geq 0$ do not induce faces of maximal dimension.*

Proof To see that the inequalities of the form $y_{ij} \geq 0$ induce facets, simply note that all but one of the affinely-independent points listed in the proof of Proposition 5 satisfy $y_{ij} = 0$. To see that the inequalities of the form $y_{ii} \geq 0$ do not induce facets, simply note that all points satisfying $y_{ii} = 0$ also satisfy $x_i = 0$. The inequalities of the form $x_i \geq 0$ are a little more tricky: one can easily construct $n + \binom{n}{2}$ affinely-independent points with $x_i = 0$, but to complete the proof one needs an additional n extreme rays of $\text{MIQ}^+_{0,n}$ having $x_i = 0$. Take one ray to have $y_{ii} = 1$ and all other variables zero, and $n-1$ rays to have $y_{ii} = y_{ij} = y_{jj} = 1$ for $j \neq i$. □

5 The integer case ($n_2 = 0$)

This section is concerned with the case in which all variables are integer-constrained, i.e., in which $n_2 = 0$.

5.1 Non-negativity inequalities

First, we consider the status of the non-negativity inequalities:

Proposition 11 *The inequalities $x_i \geq 0$ for all $1 \leq i \leq n$, and the inequalities $y_{ij} \geq 0$ for all $1 \leq i < j \leq n$, induce facets of $\text{MIQ}^+_{n,0}$. The inequalities of the form $y_{ii} \geq 0$, on the other hand, never induce facets of $\text{MIQ}^+_{n,0}$.*

Proof Just follow the proof of Theorem 3, and note that all of the affinely-independent points listed there and in the proof of Proposition 5 have integral coordinates. □

5.2 Split inequalities

It is well-known (see, e.g., [7]) that, for any vector $v \in \mathbb{Z}^n$ and scalar $s \in \mathbb{Z}$, all vectors $x \in \mathbb{Z}^n$ satisfy the so-called *split disjunction* $(v^T x \le s) \vee (v^T x \ge s+1)$. The following proposition uses split disjunctions to derive an infinite family of valid inequalities:

Proposition 12 *For any vector* $v \in \mathbb{Z}^n$ *and scalar* $s \in \mathbb{Z}$, *the following 'split' inequality is valid for both* $\text{MIQ}_{n,0}$ *and* $\text{MIQ}^+_{n,0}$*:*

$$(2s+1)v^T x + \sum_{i=1}^{n} v_i^2 y_{ii} + 2 \sum_{1 \le i < j \le n} v_i v_j y_{ij} + s(s+1) \ge 0\ . \tag{7}$$

Proof The split disjunction $(v^T x \le -s-1) \vee (v^T x \ge -s)$ implies the quadratic inequality $(v^T x + s)(v^T x + s + 1) \ge 0$. Expanding this and substituting Y for xx^T yields $v^T Y v + (2s+1)v^T x + s(s+1) \ge 0$, which is equivalent to the inequality (7). □

We remark that an important class of cutting planes for Mixed-Integer Linear Programs, called *split cuts*, can be derived using split disjunctions [7]. It is important to note however that the split inequalities (7) are *not* split cuts in the traditional sense. Indeed, split cuts arise from the interaction between a split disjunction and a set of linear constraints, whereas the split inequalities (7) are directly implied by the disjunctions themselves.

It turns out that the split inequalities dominate the psd inequalities:

Theorem 4 *The split inequalities* (7) *dominate the psd inequalities* (5).

Proof Suppose a point (x^*, y^*) violates a psd inequality with non-integral v or s, and let ϵ be a small positive quantity. Let v' be a rational vector such that $|v'_i - v_i| < \epsilon$ for all i, and let s' be a rational number such that $|s' - s| < \epsilon$. Provided ϵ is small enough, the psd inequality obtained by using v' and s' in place of v and s will also be violated by (x^*, y^*). Now let M be a positive integer such that $Mv' \in \mathbb{Z}^n$ and $Ms' \in \mathbb{Z}$. The psd inequality with Mv' and Ms' in place of v' and s' will also be violated by (x^*, y^*).

From this it follows that the psd inequalities with integral v and s define the same convex set as the general psd inequalities. (That is, even though the set of psd inequalities is uncountable, a countable subset of them suffices to describe the convex set in question.)

Now, suppose that a psd inequality is derived using an integral vector v and an integral scalar s. Recall that the psd inequality can be written as $v^T Y v + (2s)v^T x + s^2 \ge 0$. This is dominated by the two inequalities $v^T Y v + (2s+1)v^T x + s(s+1) \ge 0$ and $v^T Y v + (2s-1)v^T x + s(s-1) \ge 0$, which are both split inequalities. □

In fact, split inequalities induce facets under mild conditions:

Theorem 5 *Split inequalities induce facets of* $\text{MIQ}_{n,0}$ *if the non-zero components of* v *are relatively prime.*

Proof First, note that the trivial inequality $y_{11} \geq x_1$ is a split inequality, obtained by linearising the quadratic inequality $(x_1 - 1)x_1 \geq 0$. This trivial split inequality induces a facet of $\text{MIQ}_{n,0}$, because all but one of the affinely-independent points listed in the proof of Proposition 5 satisfy $y_{11} = x_1$.

Now consider a non-trivial split inequality of the form (7), and assume that the non-zero components of v are relatively prime. A well-known result on integral matrices (see, e.g., p. 15 of Newman [29]) implies that there exists a unimodular matrix $U \in \mathbb{Z}^{n \times n}$ having v as its first row. Let U be such a matrix, and let $w \in \mathbb{Z}^n$ be an arbitrary vector satisfying $w_1 = s + 1$. Note that, if (x, y) is an extreme point of $\text{MIQ}_{n,0}$ and (x', y') is the transformed extreme point described in Remark 2, then $x'_1 = v^T x + s + 1$ and $y'_{11} = (x'_1)^2 = v^T Y v + 2(s + 1)v^T x + (s + 1)^2$. Thus, if we apply the transformation mentioned in Corollary 3 to the trivial split inequality $y_{11} \geq x_1$, we obtain the inequality $v^T Y v + 2(s+1)v^T x + (s+1)^2 \geq v^T x + s + 1$. This is equivalent to the non-trivial split inequality. By Corollary 3, it induces a facet of $\text{MIQ}_{n,0}$. □

Theorem 6 *Split inequalities induce facets of* $\text{MIQ}^+_{n,0}$ *if the non-zero components of* v *are relatively prime and not all of the same sign.*

Proof First, note that when v satisfies the stated condition, there exists a vector $w \in \mathbb{Z}^n$ such that $v^T w = 0$ and such that $w_i > 0$ for all i. To see this, let k and k' be the number of components of v that are positive and negative, respectively, and let m be the product of the non-zero components of v. The desired vector w can be obtained by setting w_i to $k'|m|/v_i$ when $v_i > 0$, to $k|m|/|v_i|$ when $v_i < 0$, and to 1 otherwise.

Second, observe that an extreme point $(\bar{x}, \bar{y})$ of $\text{MIQ}_{n,0}$ satisfies the split inequality (7) at equality if and only if $v^T \bar{x} \in \{-s - 1, -s\}$. Therefore, if $(\bar{x}, \bar{y})$ is such an extreme point, then so is the extreme point obtained by replacing $\bar{x}$ with $\bar{x} + w$, and adjusting $\bar{y}$ accordingly. Let us call this (affine) transformation 'shifting'.

Now, since the split inequality induces a facet of $\text{MIQ}_{n,0}$ under the stated conditions, there exist $n + \binom{n+1}{2}$ affinely-independent points in $F_{n,0}$ that satisfy the split inequality at equality. By shifting this set of points, repeatedly if necessary, we obtain $n + \binom{n+1}{2}$ affinely-independent points in $F^+_{n,0}$ that satisfy the split inequality at equality. Therefore the split inequality induces a facet of $\text{MIQ}^+_{n,0}$ as well. □

Remark 3 A split inequality is satisfied at equality at the origin if and only if $s \in \{0, -1\}$. Moreover, when $n \geq 2$, there is an infinite number of vectors v satisfying the condition in either Theorems 5 or 6. It follows that, when $n \geq 2$, the origin lies on an infinite (though countable) number of facets of either $\text{MIQ}_{n,0}$ or $\text{MIQ}^+_{n,0}$. It then follows from Remark 2 that, again when $n \geq 2$, *every* extreme point of $\text{MIQ}_{n,0}$ lies on an infinite number of facets. The same can be shown for $\text{MIQ}^+_{n,0}$ (proof omitted for brevity).

If the non-zero components of the vector v all have the same sign, then the split inequality need not induce even a proper face of $\text{MIQ}^+_{n,0}$, because there may not exist a lattice point $x \in \mathbb{Z}^n_+$ such that $v^T x \in \{-s - 1, -s\}$. Theorem 2 implies however the following result:

Corollary 4 *Let* $v \in \mathbb{Z}^n$ *be such that all its components are relatively prime and of the same sign. Then there exists an integer* s *such that the split inequality* (7) *induces a facet of* $\text{MIQ}^+_{n,0}$.

Proof Let v be as stated and let s be an arbitrary integer. By Theorem 5, the corresponding split inequality defines a facet of $\text{MIQ}_{n,0}$. Then, let the vector $t \in \mathbb{Z}^n_+$ be as defined in the proof of Theorem 2. One can check that the corresponding inequality (3), which induces a facet of $\text{MIQ}^+_{n,0}$, is nothing but the split inequality that is obtained by replacing s with $s - v^T t$. □

At first sight, it may appear that the split inequalities can be generalised, as expressed in the following lemma:

Lemma 3 *For any vector $v \in \mathbb{R}^n$ and scalar $s \in \mathbb{R}$, let*

$$s^- := \sup\left\{v^T x : x \in \mathbb{Z}^n, v^T x \le s\right\}$$
$$s^+ := \inf\left\{v^T x : x \in \mathbb{Z}^n, v^T x \ge s\right\}.$$

Then, for any (u^-, u^+) satisfying $s^- \le u^- \le u^+ \le s^+$, the inequality

$$-\left(u^- + u^+\right) v^T x + \sum_{i=1}^{n} v_i^2 y_{ii} + 2 \sum_{1 \le i < j \le n} v_i v_j y_{ij} + u^- u^+ \ge 0 \tag{8}$$

is valid for both $\text{MIQ}_{n,0}$ *and* $\text{MIQ}^+_{n,0}$.

Proof Similar to Proposition 12. □

It turns out, however, that this does not yield any interesting inequalities:

Proposition 13 *Every inequality of the form (8) is either a split inequality, or dominated by split inequalities.*

Proof Without loss of generality, we can assume that the vector v is scaled so that $v_1 = 1$. Then, if any of $v_2, \ldots, v_n$ are irrational, we have $s^+ = s^- = s$ and the inequality (8) reduces to a psd inequality. The result then follows from Theorem 4.

So suppose that v is rational. We can assume that it has been scaled so that all coefficients are relatively prime integers. Then, we have $s^- = \lfloor s \rfloor$ and $s^+ = \lceil s \rceil$. For brevity, we write the inequality (8) in the 'shorthand' form $v^T Y v - (u^- + u^+) v^T x + u^- u^+ \ge 0$. Then, we distinguish two cases. If $u^- + u^+ \le s^- + s^+$, the inequality is a convex combination of the split inequalities $v^T Y v - (s^- + s^+) v^T x + s^- s^+ \ge 0$ and $v^T Y v - (2s^- - 1) v^T x + (s^- - 1) s^- \ge 0$. If on the other hand $u^- + u^+ > s^- + s^+$, it is a convex combination of the split inequalities $v^T Y v - (s^- + s^+) v^T x + s^- s^+ \ge 0$ and $v^T Y v - (2s^+ + 1) v^T x + s^+(s^+ + 1) \ge 0$. □

If, on the other hand, one imposes $x \in \mathbb{Z}^n_+$ in the definition of s^- and s^+, one can in principle obtain valid inequalities for $\text{MIQ}^+_{n,0}$ that dominate split inequalities. Inequalities of this kind, called *gap* inequalities, are studied in [12].

5.3 Inequalities from the Boolean quadric polytope

Now recall the definition of the *Boolean quadric polytope* from Sect. 2.3. The following theorem states that BQP_n is essentially nothing but a face of both $\text{MIQ}_{n,0}$ and $\text{MIQ}^+_{n,0}$:

Theorem 7 *Suppose we intersect* $\text{MIQ}_{n,0}$ *(or* $\text{MIQ}^+_{n,0}$*) with the hyperplanes defined by the following n equations:*

$$y_{ii} = x_i \quad (i = 1, \dots, n).$$

Then we obtain a face of $\text{MIQ}_{n,0}$ *(or* $\text{MIQ}^+_{n,0}$*) of dimension* $n + \binom{n}{2}$*. This face is an affine image of the Boolean quadric polytope* BQP_n*.*

Proof First, note that, for all $1 \le i \le n$, the inequality $y_{ii} \ge x_i$ is valid for $\text{MIQ}_{n,0}$. Indeed, it is a split inequality of the form (7), obtained by setting $v_i = 1$, $v_j = 0$ for all $j \ne i$, and $s = -1$. So, the intersection of $\text{MIQ}_{n,0}$ and the specified hyperplanes is indeed a face of $\text{MIQ}_{n,0}$. Let H denote this face.

Now, note that an extreme point of $\text{MIQ}_{n,0}$ satisfies $y_{ii} = x_i$, for some i, if and only if it satisfies $x_i \in \{0, 1\}$. Therefore, the extreme points of H are precisely the members of $F_{n,0}$ that satisfy $x \in \{0, 1\}^n$. So, there is a one-to-one correspondence between extreme points of H and extreme points of BQP_n. Moreover, every extreme point (x^*, y^*) of BQP_n can be mapped onto an extreme point of H simply by setting $y^*_{ii} = x^*_i$ for all $i = 1, \dots, n$. This mapping is affine and dimension-preserving.

The proof for $\text{MIQ}^+_{n,0}$ is identical. □

Theorem 7 has the following useful corollary:

Corollary 5 *Suppose the inequality*

$$\sum_{i=1}^{n} a_i x_i + \sum_{1 \le i < j \le n} b_{ij} y_{ij} \le c$$

induces a facet of BQP_n*. Then there exists at least one 'lifted' inequality of the form*

$$\sum_{i=1}^{n} (a_i - \lambda_i) x_i + \sum_{i=1}^{n} \lambda_i y_{ii} + \sum_{1 \le i < j \le n} b_{ij} y_{ij} \le c\,,$$

with $\lambda \in \mathbb{Q}^n$*, that induces a facet of* $\text{MIQ}_{n,0}$*, and similarly for* $\text{MIQ}^+_{n,0}$.

To illustrate Corollary 5, we apply it to the following inequality:

$$\sum_{i=1}^{5} y_{i6} \le 2x_6 + y_{12} + y_{23} + y_{34} + y_{45} + y_{15}. \qquad (9)$$

One can easily check (either by hand or with the aid of a computer) the following facts:

- The inequality (9) induces a facet of BQP_6.
- It is valid also for $\text{MIQ}^+_{6,0}$, and induces an unbounded facet of it. (To see that it is unbounded, observe that, for any $1 \le i \le 5$ and any positive integer t, we can obtain a member of $F^+_{6,0}$ lying on the facet by setting x_i to t, y_{ii} to t^2, and all other variables to zero.)

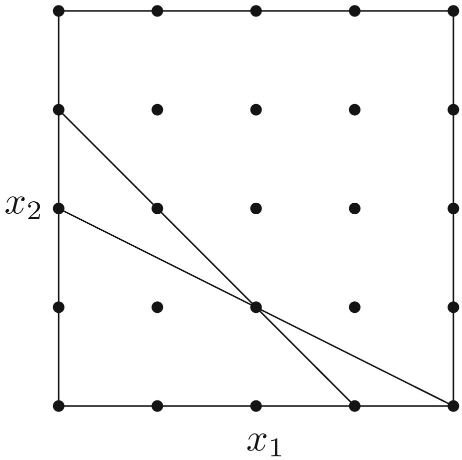

Fig. 2 A 'non-standard' split when $n = 2$

- It is not valid for $\mathrm{MIQ}_{6,0}$, but the lifted version

$$\sum_{i=1}^{5} y_{i6} \leq 2x_6 + y_{12} + y_{23} + y_{34} + y_{45} + y_{15} + \sum_{i=1}^{6}(y_{ii} - x_{ii}) \tag{10}$$

  is valid for $\mathrm{MIQ}_{6,0}$, and induces a bounded facet of it.

We observed an interesting feature of the lifted inequality (10). There are 27 extreme points of $\mathrm{MIQ}_{6,0}$ that satisfy it at equality. If we take the corresponding 27 points in x-space, then their convex hull turns out to be an affine image of a famous polytope in the theory of Delaunay polytopes (see [8]); namely, the 6-dimensional polytope of Gosset [13]. (For reasons of space, we do not give a formal proof of this fact.) We suspect that this is not a coincidence, and that there is some deep connection between facets of $\mathrm{MIQ}_{n,0}$ and Delaunay polytopes. This issue is left for future research.

5.4 Inequalities for $\mathrm{MIQ}^+_{n,0}$ from non-standard splits

To close this section, we point out that, when $n \geq 2$, one can derive further facet-inducing inequalities for $\mathrm{MIQ}^+_{n,0}$ using a 'non-standard' split disjunction.

Consider the two lines in $\mathbb{R}^2$ defined by the equations $x_1+x_2 = 3$ and $x_1+2x_2 = 4$. As illustrated in Fig. 2, these lines pass through several points in $\mathbb{Z}^2_+$. Moreover, all points in $\mathbb{Z}^2_+$ are either above both lines (satisfying $x_1 + x_2 \geq 3$ and $x_1 + 2x_2 \geq 4$), or below both lines (satisfying $x_1 + x_2 \leq 3$ and $x_1 + 2x_2 \leq 4$). This implies that all points in $F^+_{2,0}$ satisfy the non-linear inequality

$$(x_1 + x_2 - 3)(x_1 + 2x_2 - 4) \geq 0.$$

This in turn implies that the linear inequality

$$-7x_1 - 9x_2 + y_{11} + 3y_{12} + 2y_{22} \geq 12$$

is valid for $\text{MIQ}^+_{2,0}$. One can check (either by hand or with the aid of a computer) that this inequality induces a facet of $\text{MIQ}^+_{2,0}$.

One can easily derive other valid inequalities of a similar kind for $\text{MIQ}^+_{2,0}$, or indeed for $\text{MIQ}^+_{n,0}$ with $n > 2$. We leave for future research the task of characterising the non-standard split disjunctions that lead to inequalities inducing facets of $\text{MIQ}^+_{n,0}$.

6 The mixed-integer case ($n_1 > 0$ and $n_2 > 0$)

Now we move on to the more general *mixed* case, in which both n_1 and n_2 are permitted to be positive.

6.1 Canonical extension

One easy way to adapt results for the pure integer case to the mixed case is to use the following simple observation. If the linear inequality

$$\sum_{i=1}^{n_1} a_i x_i + \sum_{1 \leq i \leq j \leq n_1} b_{ij} y_{ij} \leq c \tag{11}$$

is valid for $\text{MIQ}_{n_1,0}$, then it is also valid for MIQ_{n_1,n_2}. Similarly, if it is valid for $\text{MIQ}^+_{n_1,0}$, then it is also valid for $\text{MIQ}^+_{n_1,n_2}$. Padberg [30] used a similar operation in the context of the Boolean quadric polytope, calling it 'canonical extension'. We also used it in [6].

One can also use canonical extension to adapt results for the continuous case to the mixed case. Namely, if the linear inequality

$$\sum_{i=1}^{n_2} a_i x_i + \sum_{1 \leq i \leq j \leq n_2} b_{ij} y_{ij} \leq c \tag{12}$$

is valid for MIQ_{0,n_2}, then the inequality

$$\sum_{i=n_1+1}^{n} a_i x_i + \sum_{n_1+1 \leq i \leq j \leq n} b_{ij} y_{ij} \leq c \tag{13}$$

is valid for MIQ_{n_1,n_2}.

Now, as in [6], we say that a face of a p-dimensional convex body has *co-dimension* k if the face has dimension $p - k$. (For example, the co-dimension of a facet is 1, and the co-dimension of a psd inequality for MIQ_{0,n_2} or MIQ_{0,n_2} is at least $n + 1$.)

It turns out that canonical extension preserves co-dimension under mild conditions. This is made precise in the following two propositions:

Proposition 14 *Suppose that the linear inequality* (11) *induces a face of* $\mathrm{MIQ}_{n_1,0}$ *of co-dimension k, where* $1 \le k \le n_1$*. Then it also induces a face of* MIQ_{n_1,n_2} *of co-dimension k, for all* $n_2 \ge 1$*. Moreover, the analogous statement holds for* $\mathrm{MIQ}^+_{n_1,0}$ *and* $\mathrm{MIQ}^+_{n_1,n_2}$*.*

Proposition 15 *Suppose that the linear inequality* (12) *induces a face of* MIQ_{0,n_2} *of co-dimension k, where* $1 \le k \le n_1$*. Then the inequality* (13) *induces a face of* MIQ_{n_1,n_2} *of co-dimension k, for all* $n_1 \ge 1$*. Moreover, the analogous statement holds for* MIQ^+_{0,n_2} *and* $\mathrm{MIQ}^+_{n_1,n_2}$*.*

For the sake of brevity, we omit detailed proofs of these two propositions. The proofs are similar to that of Theorem 3 in [6], the only difference being that one has to deal with extreme rays as well as extreme points, due to the fact that MIQ_{n_1,n_2} and $\mathrm{MIQ}^+_{n_1,n_2}$ are unbounded.

6.2 Non-negativity inequalities

The results of the previous subsection enable us to quickly settle the status of the non-negativity inequalities:

Corollary 6 *The inequalities* $x_i \ge 0$ *for all* $1 \le i \le n$*, and the inequalities* $y_{ij} \ge 0$ *for all* $1 \le i < j \le n$*, induce facets of* $\mathrm{MIQ}^+_{n_1,n_2}$*, for all* $n_1 \ge 1$ *and* $n_2 \ge 1$*. The inequalities of the form* $y_{ii} \ge 0$ *never induce faces of maximal dimension.*

Proof This follows from Theorem 3 and Propositions 11, 14 and 15. □

6.3 Split inequalities

Next, we examine the status of the split inequalities (7) in the mixed case.

First, notice that the split disjunction $(v^T x \le s) \vee (v^T x \ge s+1)$, with integral v and s, is valid for F_{n_1,n_2} if and only if it does not involve any continuous variables, i.e., if and only if $v_i = 0$ for $i = n_1+1, \ldots, n$. As a result, a split inequality is valid for MIQ_{n_1,n_2} if and only if it is the canonical extension of a split inequality for $\mathrm{MIQ}_{n_1,0}$.

The situation with $\mathrm{MIQ}^+_{n_1,n_2}$ is a bit more subtle. It is true that a split disjunction that does not involve any continuous variables is valid for $F^+_{n_1,n_2}$, but this condition is no longer necessary. For example, if x_i is any continuous and non-negative variable, the disjunction $(x_i \le -1) \vee (x_i \ge 0)$ is (trivially) valid for $F^+_{n_1,n_2}$. We conjecture, however, that split disjunctions that do not meet the condition can never lead to facet-defining split inequalities.

In any case, Propositions 14 and 15 imply the following result:

Corollary 7 *Consider a facet-defining inequality of* $\mathrm{MIQ}_{n_1,0}$ *or* $\mathrm{MIQ}^+_{n_1,0}$ *as described in Theorem* 5 *or* 6*. Its canonical extension induces a facet of* MIQ_{n_1,n_2} *or* $\mathrm{MIQ}^+_{n_1,n_2}$ *for all* $n_2 \ge 1$*.*

6.4 Psd inequalities

Finally, we consider the psd inequalities (5). The following two propositions settle most cases:

Proposition 16 *Suppose that a psd inequality involves at least one continuous variable, i.e., that $v_i \neq 0$ for some $n_1 < i \leq n$. Then it induces a face of* MIQ_{n_1,n_2} *of maximal dimension, and the dimension is* $\binom{n+1}{2} - 1$.

Proof Let v^1 be the first n_1 components of v, and let v^2 be the last n_2 components. Then $v^2 \neq 0$, and we assume without loss of generality that $v_n = v^2_{n_2} \neq 0$.

Proposition 5 establishes that the dimension of MIQ_{n_1,n_2-1} is $n - 1 + \binom{n}{2} = \binom{n+1}{2} - 1$. In particular, its proof demonstrates $\binom{n+1}{2}$ affinely independent extreme points, each of the form (x, xx^T). Because $v_n \neq 0$, it is easy to extend each such (x, xx^T) to an extreme point $(\bar{x}, \bar{x}\bar{x}^T)$ of MIQ_{n_1,n_2} satisfying $v^T\bar{x} + s = 0$ and hence lying on the face. The resulting extreme points remain affinely independent. So the dimension of the face is at least $\binom{n+1}{2} - 1$.

Now, we know from Proposition 9 that the face of $\text{MIQ}_{0,n}$ induced by the psd inequality has dimension $\binom{n+1}{2} - 1$. Since $\text{MIQ}_{n_1,n_2} \subseteq \text{MIQ}_{0,n}$, the face of MIQ_{n_1,n_2} cannot have larger dimension. □

Proposition 17 *If a psd inequality does not involve any continuous variables, i.e., if $v_i = 0$ for $n_1 < i \leq n$, then it does not induce a face of maximal dimension for either* MIQ_{n_1,n_2} *or* $\text{MIQ}^+_{n_1,n_2}$.

Proof Under the stated condition, the psd inequality is the canonical extension of a psd inequality for both $\text{MIQ}_{n_1,0}$ and $\text{MIQ}^+_{n_1,0}$. It then follows from Theorem 4 and Propositions 14 and 15 that the original psd inequality is dominated by the canonical extensions of split inequalities for $\text{MIQ}_{n_1,0}$ and $\text{MIQ}^+_{n_1,0}$. □

The remaining case is covered in the following proposition.

Proposition 18 *Suppose that a psd inequality involves at least one continuous variable, i.e., that $v_i \neq 0$ for some $n_1 < i \leq n$. If, in addition, not all non-zero components of v have the same sign, then the inequality induces a face of* $\text{MIQ}^+_{n_1,n_2}$ *of maximal dimension, and the dimension is* $\binom{n+1}{2} - 1$. *If all non-zero components of v have the same sign, then the inequality may or may not induce a face of maximal dimension.*

This can be proved by combining the proof of Proposition 16 with the 'shifting' operation described in the proof of Theorem 6. We omit further details for the sake of brevity.

7 Complete linear descriptions

In this last main section of the paper, we discuss complete linear descriptions for MIQ_{n_1,n_2} and $\text{MIQ}^+_{n_1,n_2}$ for small n.

The continuous case is straightforward. Proposition 9 states that $\text{MIQ}_{0,n}$ is completely described by psd inequalities, for all n. On the other hand, $\text{MIQ}^+_{0,n}$ is completely described by psd and non-negativity inequalities if and only if $n \leq 3$. (This follows from Proposition 8, together with the fact, from Maxfield and Minc [24], that the set of completely positive matrices is equal to the set of doubly non-negative matrices if and only if $n \leq 4$.) In particular, one sees that $\text{MIQ}_{0,1}$ is also described by the single convex quadratic inequality $y_{11} \geq x_1^2$, and that $\text{MIQ}^+_{0,1}$ is described by the convex quadratic inequality $y_{11} \geq x_1^2$ and the non-negativity inequality $x_1 \geq 0$.

The pure integer case is also straightforward when $n = 1$. From Fig. 1, one sees that $\text{MIQ}^+_{1,0}$ is described by the non-negativity inequality $x_1 \geq 0$, together with the split inequalities $y_{11} \geq (2t + 1)x_1 - t(t + 1)$ for all $t \in \mathbb{Z}_+$. A similar observation was made in [27] for a related family of polytopes. One can also check that $\text{MIQ}_{1,0}$ is described by split inequalities of the same form, but for all $t \in \mathbb{Z}$.

Now, we saw in Sect. 5.4 that the split and non-negativity inequalities are not sufficient to describe $\text{MIQ}^+_{2,0}$. A natural question is whether the split inequalities are enough to describe $\text{MIQ}_{2,0}$. We show that this is indeed true, but, as the proof is quite involved, we first introduce some notation and two lemmas to simplify the proof.

We will represent a general valid inequality for $\text{MIQ}_{n,0}$ as $A \bullet Y + 2b^T x + \gamma \geq 0$ for some symmetric matrix A, vector b, and scalar γ, where $A \bullet Y := \text{trace}(AY)$. Since validity of $A \bullet Y + 2b^T x + \gamma \geq 0$ is equivalent to validity of its quadratic counterpart $x^T Ax + 2b^T x + \gamma \geq 0$ over $\mathbb{Z}^n$, we will switch back and forth without comment as convenient.

For any v, s, the psd inequality (5) can be written in the form $A \bullet Y + 2b^T x + \gamma \geq 0$ with $A := vv^T$, $b := sv$, and $\gamma := s^2$; the proof of Lemma 2 provides insight into this representation. In particular, A is rank-1 psd in this case, and a partial converse holds:

Lemma 4 *Suppose $A \bullet Y + 2b^T x + \gamma \geq 0$ is valid for $\text{MIQ}_{n,0}$. Then A is psd.*

Proof Suppose A is not psd, and let w be a negative eigenvector of A. There exists a nearby rational vector w' such that $(w')^T Aw' < 0$, and so there exists $M > 0$ with $u := Mw' \in \mathbb{Z}^n$ and $u^T Au < 0$. Then for large integer $k > 0$, we have $(ku)^T A(ku) + 2b^T(ku) + \gamma = k^2 \cdot u^T Au + k \cdot 2b^T u + \gamma < 0$. This proves $A \bullet Y + 2b^T x + \gamma \geq 0$ is not valid for $\text{MIQ}_{n,0}$. □

We will also use the following lemma, which provides conditions under which a particular valid inequality is dominated.

Lemma 5 *Suppose $q(x) := x^T Ax + 2b^T x + \gamma \geq 0$ and $r(x) := x^T Bx + 2c^T x + \delta \geq 0$ are valid over $\mathbb{Z}^n$. Suppose also that A is positive definite and $r(x) = 0$ holds whenever $q(x) = 0$ and $x \in \mathbb{Z}^n$. Then there exists $\epsilon > 0$ such that $q(x) - \epsilon r(x) \geq 0$ is valid over $x \in \mathbb{Z}^n$. In particular, $q(x)$ is dominated by $r(x) \geq 0$ and $q(x) - \epsilon r(x) \geq 0$.*

Proof Let $\bar{\epsilon} > 0$ be such that $A - \bar{\epsilon}B \succ 0$. Then, because $A - \bar{\epsilon}B$ is the Hessian of $q(x) - \epsilon r(x)$, there exists a radius $r > 0$ such that $q(x) - \epsilon r(x) \geq 0$ is valid on $\{x \in \mathbb{Z}^n : \|x\| > r\}$ for all $\epsilon \leq \bar{\epsilon}$. On the other hand, it is easy to see the existence of $\hat{\epsilon} > 0$ such $q(x) - \epsilon r(x) \geq 0$ is valid on the finite set $\{x \in \mathbb{Z}^n : \|x\| \leq r\}$ for all

$\epsilon \leq \hat{\epsilon}$ because $x^T Ax + 2b^T x + \gamma = 0$ implies $x^T Bx + 2c^T x + \delta = 0$. Now simply take $\epsilon = \min\{\bar{\epsilon}, \hat{\epsilon}\} > 0$. □

We are now ready to show that the split inequalities are enough to capture $\text{MIQ}_{2,0}$.

Theorem 8 $\text{MIQ}_{2,0}$ *is completely described by the split inequalities.*

Proof Consider any valid inequality $A \bullet Y + 2b^T x + \gamma \geq 0$ for $\text{MIQ}_{2,0}$. If $A = 0$, then $b^T x + \gamma \geq 0$ is valid if and only if $b = 0$ and $\gamma \geq 0$. So we have a nonnegative multiple of the split inequality arising from $v = 0$ and $s = 1$.

So assume $A \neq 0$. Define $\mu := \inf\{x^T Ax + 2b^T x + \gamma : x \in \mathbb{Z}^n\}$, and note that $A \bullet Y + 2b^T x + \gamma \geq 0$ is dominated by the valid inequality $A \bullet Y + 2b^T x + \gamma - \mu \geq 0$. Moreover, the corresponding infimum for this new inequality is 0. So we may reduce to the case $\mu = 0$. Lemma 4 implies $A \succeq 0$. Since A is 2×2, there are only two possibilities for the rank of A: either $\text{rank}(A) = 1$ or $\text{rank}(A) = 2$.

Suppose $\text{rank}(A) = 1$ and write $A = aa^T$. Since $x^T Ax + 2b^T x + \gamma$ is bounded below on $\mathbb{Z}^n$, it is bounded below on $\mathbb{R}^n$. Hence, by the Frank–Wolfe theorem, $x^T Ax + 2b^T x + \gamma$ attains its minimum on $\mathbb{R}^n$ with first-order conditions $0 = Ax + b = (a^T x)a + b$. In particular, $b = \rho a$ for some $\rho \in \mathbb{R}$. Our valid inequality can then be written $a^T Ya + 2\rho a^T x + \gamma \geq 0$, and we show that this is precisely a "generalised" split inequality (8) based on a pair (v, s). Specifically, take $(v, s) = (a, -\rho)$ and define s^- and s^+ as in Lemma 3. Now suppose s^+ is closer to s than s^- (the other case is similar). Writing $\epsilon := s^+ - s$, we take $(u^-, u^+) = (s - \epsilon, s + \epsilon)$ and have via (8) the generalised split inequality

$$v^T Yv - ((s - \epsilon) + (s + \epsilon))v^T x + (s - \epsilon)(s + \epsilon) \geq 0,$$

which simplifies to $a^T Ya + 2\rho a^T x + \rho^2 - \epsilon^2 \geq 0$ since $(v, s) = (a, -\rho)$. This matches our valid inequality in all coefficients except possibly the constant term (γ versus $\rho^2 - \epsilon^2$). However, by construction, the split inequality has infimum 0 over $\mathbb{Z}^n$. So does our valid inequality. It follows that $\gamma = \rho^2 - \epsilon^2$ and the two inequalities are indeed the same. Proposition 13 now implies that $A \bullet Y + 2b^T x + \gamma \geq 0$ is dominated by split inequalities.

Finally, suppose $\text{rank}(A) = 2$, which implies $q(x) := x^T Ax + 2b^T x + \gamma$ has ellipsoidal level sets. Define $Z := \{z \in \mathbb{Z}^2 : q(z) = 0\}$, and because $\mu = 0$ and $q(x)$ has compact level sets, we have that $|Z| \geq 1$ and Z is contained in the boundary of an ellipsoid. In particular, no $z \in Z$ can be expressed as a proper convex combination of other points in Z. So $1 \leq |Z| \leq 2$, or the convex hull of Z is a polygon with $|Z|$ edges. Also, from Arkinstall [2], Lovász [22], Rabinowitz [32] we know that any polygon with integer vertices and 5 or more edges must contain an integer point in its interior. Thus, we have the following cases: (i) $1 \leq |Z| \leq 2$; (ii) $3 \leq |Z| \leq 4$.

Suppose $1 \leq |Z| \leq 2$. Without loss of generality, by an affine unimodular transformation, we may assume Z contains 0. If Z contains a second member (z_1, z_2), set $v = (-z_2, z_1)$; otherwise, set $v \in \mathbb{Z}^2$ arbitrarily. Consider the split inequality $r(x) := v^T x(v^T x + 1) \geq 0$, and note that the zeros of the split inequality contain Z. Lemma 5 then shows that $q(x)$ is dominated.

Now suppose $3 \leq |Z| \leq 4$. Since the convex hull of Z is a polygon with no interior integer points, the papers [2,22,32] prove that—modulo a unimodular affine transformation, which does not alter the validity of $A \bullet Y + 2b^T x + \gamma$ by Corollary 3—Z contains either the points $Z_p := \{(0,0), (p,0), (0,1)\}$ for some integer $p > 0$ or $Z_2 := \{(0,0), (2,0), (0,2)\}$. In fact, we claim that Z must contain Z_p by supposing $Z_2 \subseteq Z$ and deriving a contradiction. Since $q(z) = 0$ for all $z \in Z_2$, one sees immediately that $\gamma = 0$, $b_1 = -A_{11}$, and $b_2 = -A_{22}$. Hence,

$$q(x) = A_{11}\left(x_1^2 - 2x_1\right) + A_{22}\left(x_2^2 - 2x_2\right) + 2A_{21}x_1x_2,$$

which implies in particular that $q(1,0) = -A_{11}$. Since $A_{11} > 0$ because $\operatorname{rank}(A) = 2$, this shows $q(x)$ attains a negative value on $\mathbb{Z}^2$, which is the desired contradiction. So $Z_p \subseteq Z$ in which case we deduce similarly that

$$q(x) = A_{11}\left(x_1^2 - px_1\right) + A_{22}\left(x_2^2 - x_2\right) + 2A_{21}x_1x_2.$$

Since $A_{11} > 0$ and $q(1,0) \geq 0$, we have $A_{11}(1-p) \geq 0 \Leftrightarrow p = 1$. Also, $q(1,1) \geq 0$ implies $A_{21} \geq 0$, and $q(-1,1) \geq 0$ and $q(1,-1) \geq 0$ imply $A_{21} \leq \min\{A_{11}, A_{22}\}$. So we may write

$$\begin{aligned} q(x) = (A_{11} - A_{21})\left(x_1^2 - x_1\right) + (A_{22} - A_{21})\left(x_2^2 - x_2\right) \\ + A_{21}\left[\left(x_1^2 - x_1\right) + \left(x_2^2 - x_2\right) + 2x_1x_2\right] \end{aligned}$$

with $A_{11} - A_{21} \geq 0$, $A_{22} - A_{21} \geq 0$, and $A_{21} \geq 0$. So $q(x)$ is the nonnegative combination of three quadratics, each of which clearly corresponds to a split inequality. So $A \bullet Y + 2b^T x + \gamma \geq 0$ is dominated by split inequalities. □

We do not know if the split inequalities suffice to capture $\mathrm{MIQ}_{n,0}$ for some $n > 2$. On the other hand, the inequality (10) is not a split inequality, yet induces a facet of $\mathrm{MIQ}_{6,0}$. This shows that the split inequalities do not completely describe $\mathrm{MIQ}_{6,0}$.

Finally, the mixed case appears even more difficult. We do not know whether psd and split inequalities are enough to describe $\mathrm{MIQ}_{1,1}$, nor whether psd, split and non-negativity inequalities are enough to describe $\mathrm{MIQ}^+_{1,1}$.

8 Concluding remarks

In this paper, we have proved various results for the convex sets associated with unconstrained non-convex Mixed-Integer Quadratic Programs. It is our hope that the valid inequalities that we have derived will be used as cutting planes within exact algorithms for non-convex MIQPs, whether constrained or not. As mentioned in the introduction, they could be used for problems with quadratic constraints as well.

There are many interesting open theoretical questions. We have already mentioned the question of whether one can optimise a linear function over $\mathrm{MIQ}^+_{n,0}$ in polynomial time for fixed n (Sect. 3.2), the problem of characterising the non-dominated inequalities coming from 'non-standard' splits (Sect. 5.4), and the problem of finding complete linear descriptions of MIQ_{n_1,n_2} and $\mathrm{MIQ}^+_{n_1,n_2}$ for certain small values of n_1 and n_2 (Sect. 7).

Another important question is whether the separation problem for the split inequalities (7) can be solved in polynomial time. That is, whether one can efficiently find a split inequality violated by a given pair (x^*, y^*), if one exists. Unfortunately, we conjecture that this problem is strongly $\mathcal{NP}$-hard. On the other hand, the separation problem for the weaker psd inequalities (5) can be easily solved in polynomial time by computing the minimum eigenvalue of the matrix

$$\begin{pmatrix} 1 & (x^*)^T \\ x^* & X^* \end{pmatrix},$$

where X^* is the symmetric matrix corresponding to y^*. Perhaps an effective separation heuristic for split inequalities could be devised based on this fact. (We remark that the eigenvectors of this matrix were recently used in [34], but to derive *disjunctive* cuts for mixed-integer quadratic problems, rather than split inequalities.)

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