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FULL LENGTH PAPER

The trust region subproblem with non-intersecting linear constraints

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Abstract This paper studies an extended trust region subproblem (eTRS) in which the trust region intersects the unit ball with m linear inequality constraints. When $m = 0$, $m = 1$, or $m = 2$ and the linear constraints are parallel, it is known that the eTRS optimal value equals the optimal value of a particular convex relaxation, which is solvable in polynomial time. However, it is also known that, when $m \geq 2$ and at least two of the linear constraints intersect within the ball, i.e., some feasible point of the eTRS satisfies both linear constraints at equality, then the same convex relaxation may admit a gap with eTRS. This paper shows that the convex relaxation has no gap for arbitrary m as long as the linear constraints are non-intersecting.

Keywords Trust-region subproblem · Second-order cone programming · Semidefinite programming · Nonconvex quadratic programming

Mathematics Subject Classification 90C20 · 90C22 · 90C25 · 90C26 · 90C30

1 Introduction

The classical *trust region subproblem* minimizes a nonconvex quadratic objective over the unit ball:

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$$v(T_0) := \min_x \left\{ x^T Qx + c^T x : \|x\| \leq 1 \right\}. \qquad (T_0)$$

(T_0) is an important subproblem in trust region methods for nonlinear optimization and has drawn intense research interest [4–7,10,13]. In particular, even though (T_0) is nonconvex, it can be solved efficiently both in theory and practice. Several extensions of (T_0) have been proposed that enforce additional constraints on the trust region, e.g., parallel linear constraints [14], or a second full-dimensional ellipsoidal constraint [3]. The paper [9] replaces $\|x\| \leq 1$ with the more general constraint $l \leq q(x) \leq u$, where $q(x)$ is an arbitrary quadratic function. An important theoretical and practical issue is whether such extensions can still be solved efficiently. In this paper, we investigate the theoretical tractability of the following extension of (T_0), which enforces m additional linear inequality constraints:

$$v(T_m) \;:=\; \min_x \left\{ x^T Qx + c^T x \;:\; \begin{array}{c} \|x\| \leq 1 \\ a_i^T x \leq b_i \;\; (i = 1, \ldots, m) \end{array} \right\}. \qquad (T_m)$$

A natural starting point is of course (T_0), which has the following polynomial-time solvable semidefinite programming (SDP) relaxation:

$$v(T_0) \geq v(R_0) := \min_{x,X} \left\{ Q \bullet X + c^T x : \mathrm{trace}(X) \leq 1, \;\; X \succeq xx^T \right\}. \qquad (R_0)$$

Here, X is a symmetric matrix, $Q \bullet X$ is the matrix inner product, $X \succeq xx^T$ is equivalent to the convex constraint

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0$$

by the Schur complement theorem, and the constraint $\mathrm{trace}(X) \leq 1$ makes the feasible region of (R_0) compact. Let $\mathcal{F}(R_0)$ denote the feasible region of (R_0). One can show—by appealing to Barvinok's or Pataki's theories concerning the rank of extreme points of semidefinite feasibility systems [1,8], for example—that every extreme point of $\mathcal{F}(R_0)$ satisfies $X = xx^T$, which guarantees that $v(R_0)$ actually equals $v(T_0)$ since some optimal solution of (R_0) must occur at an extreme point.

Sturm and Zhang [12] and Burer and Anstreicher [2] study the following polynomial-time solvable relaxation of (T_1), in this case (R_0) with an added second-order cone (SOC) constraint:

$$v(T_1) \geq v(R_1) := \min_{x,X} \left\{ Q \bullet X + c^T x : \begin{array}{c} \mathrm{trace}(X) \leq 1, \;\; X \succeq xx^T \\ \|b_1 x - X a_1\| \leq b_1 - a_1^T x \end{array} \right\}. \qquad (R_1)$$

The SOC constraint is constructed [12] by relaxing the valid quadratic SOC constraint $\|(b_1 - a_1^T x)x\| = (b_1 - a_1^T x)\|x\| \leq b_1 - a_1^T x$ and is called an *SOC-RLT constraint* [2] since its construction is closely related to the reformulation-linearization technique of [11]. Sturm and Zhang prove $v(T_1) = v(R_1)$, extending the case for $m = 0$, while Burer and Anstreicher show further that every extreme point of $\mathcal{F}(R_1)$ satisfies $X = xx^T$, where $\mathcal{F}(R_1)$ is the feasible set of (R_1).

Ye and Zhang [14] and Burer and Anstreicher [2] also studied (T_2), which has the following polynomial-time solvable relaxation:

$$v(T_2) \geq v(R_2) := \min_{x,X} \left\{ Q \bullet X + c^T x : \begin{array}{c} \operatorname{trace}(X) \leq 1,\ X \succeq xx^T \\ \|b_i x - X a_i\| \leq b_i - a_i^T x \ (i = 1, 2) \\ b_1 b_2 - b_2\, a_1^T x - b_1\, a_2^T x + a_1^T X a_2 \geq 0 \end{array} \right\}. \quad (R_2)$$

Extending (R_1), this relaxation also contains the SOC-RLT constraint for $a_2^T x \leq b_2$ and a new linear *RLT* constraint reflecting the valid quadratic inequality $(b_1 - a_1^T x)(b_2 - a_2^T x) \geq 0$.

Ye and Zhang first showed that if the complementarity condition $(b_1 - a_1^T x)(b_2 - a_2^T x) = 0$ is added to (T_2), then the corresponding relaxation with $b_1 b_2 - b_2 a_1^T x - b_1 a_2^T x + a_1^T X a_2 = 0$ admits no gap. Based on this complementarity result, the authors then showed that the case of (T_2) with $a_1 \parallel a_2$ could be solved in polynomial-time via several steps. Burer and Anstreicher also considered $a_1 \parallel a_2$ and extended the results of Ye and Zhang by showing that the extreme points of $\mathcal{F}(R_2)$ satisfy $X = xx^T$. So $v(T_2) = v(R_2)$ in this case, thus providing a second proof that this special case of (T_2) is polynomial-time solvable—but this time in a single step.

On the other hand, Burer and Anstreicher gave a counter-example for which $v(T_2) > v(R_2)$ when $a_1 \nparallel a_2$. This example had the property that $a_1^T x \leq b_1$ and $a_2^T x \leq b_2$ intersected inside the unit ball—more precisely, some x in the unit ball simultaneously made both linear constraints tight—leaving open the possibility that $v(T_2) = v(R_2)$ could still hold when the two inequalities are non-intersecting.

In this paper, we study the following polynomial-time solvable relaxation of (T_m), which includes all possible SOC-RLT and RLT constraints:

$$\begin{aligned} v(T_m) \geq v(R_m) := \min_{x,X} \quad & Q \bullet X + c^T x \\ \text{s.t. } & \operatorname{trace}(X) \leq 1,\ X \succeq xx^T && (R_m) \\ & \|b_i x - X a_i\| \leq b_i - a_i^T x && 1 \leq i \leq m \\ & b_i b_j - b_j a_i^T x - b_i a_j^T x + a_i^T X a_j \geq 0 && 1 \leq i < j \leq m \end{aligned}$$

In light of the results in [2], we focus only on the non-intersecting case, that is, when there exists no x feasible for (T_m) satisfying $a_i^T x = b_i$ and $a_j^T x = b_j$ for some $i < j$. Note that the RLT constraints corresponding to $i = j$ are implied by $X \succeq xx^T$, and by the symmetry of X, the RLT constraint for $i > j$ is equivalent to the RLT constraint for $(\hat{\imath}, \hat{\jmath}) = (j, i)$ with $\hat{\imath} < \hat{\jmath}$.

Let $\mathcal{F}(R_m)$ denote the feasible set of (R_m). For the non-intersecting case, we will prove that every extreme point of $\mathcal{F}(R_m)$ satisfies $X = xx^T$, which immediately implies $v(T_m) = v(R_m)$ and that (T_m) is polynomial-time solvable. This is our main goal. Combined with the results of [2], we thus achieve a very clear demarcation of when the relaxation (R_m) achieves $v(T_m) = v(R_m)$ generally: precisely when the linear constraints $a_i^T x \le b_i$ are non-intersecting in the unit ball. We also discuss a slight extension in the last section of the paper.

2 Ye and Zhang's analysis revisited

As motivation for our result, we would first like to discuss how the original analyis of Ye and Zhang [14], which solves the case of (T_2) with $a_1 \parallel a_2$ in several polynomial-time steps, extends naturally to the non-intersecting case of this paper. In other words, the same steps solve the non-intersecting case of (T_2) in polynomial time. Thus, at least for $m = 2$, our result may not be surprising. On the other hand, we could not see how to extend the approach of Ye and Zhang to general m, and so to the best of our knowledge, our general approach in this paper contributes something new.

In Sect. 2.3 of their paper [14], Ye and Zhang proved that

$$v(T_2^c) \;:=\; \min_x \left\{ x^T Q x + c^T x \;:\; \begin{array}{c} \|x\| \le 1 \\ a_i^T x \le b_i \;(i = 1, 2) \\ (b_1 - a_1^T x)(b_2 - a_2^T x) = 0 \end{array} \right\} \qquad (T_2^c)$$

is solved exactly by the SOCP-SDP relaxation

$$v(R_2^c) := \min_{x,X} \left\{ Q \bullet X + c^T x : \begin{array}{c} \mathrm{trace}(X) \le 1,\; X \succeq xx^T \\ \|b_i x - X a_i\| \le b_i - a_i^T x \;(i = 1, 2) \\ b_1 b_2 - b_2\, a_1^T x - b_1\, a_2^T x + a_1^T X a_2 = 0 \end{array} \right\}. \qquad (R_2^c)$$

In words, $v(R_2^c) = v(T_2^c)$.

In Sect. 4 of [14], Ye and Zhang then analyzed the following specific case of (T_2), which is essentially equivalent to (T_2) with $a_1 \parallel a_2$:

$$\min_x \left\{ x^T Q x + c^T x \;:\; \begin{array}{c} \|x\| \le 1 \\ -1 \le a_0^T x - b_0 \le 1 \end{array} \right\}. \qquad (1)$$

They argued that it could be solved in two steps. First, assume that either $-1 \le a_0^T x - b_0$ or $a_0^T x - b_0 \le 1$ is binding at optimality. Then $(a_0^T x - b_0 + 1)(1 - a_0^T x + b_0) = 0$ is valid at optimality, and hence solving an instance of (R_2^c) recovers the optimal value. Second, if $-1 < a_0^T x - b_0 < 1$ at optimality, then the various candidates for the optimal solution of (T_0), when both linear constraints are dropped, can be examined. From the theory known for (T_0), these candidates are solutions of

$$(Q + \mu I)x = -\tfrac{1}{2}c, \qquad \|x\| = 1,$$

where μ varies over a list of three potential optimal values that can be pre-computed separately. However, it is not enough for a candidate x to be optimal for (T_0). It must also be feasible, i.e., it must satisfy

$$(Q + \mu I)x = -\tfrac{1}{2}c, \qquad \|x\| = 1, \qquad -1 \le a_0^T x - b_0 \le 1$$

simultaneously. To find such candidates, Ye and Zhang suggest to solve

$$\max_x \left\{ \left(a_0^T x - b_0 + 1\right)\left(1 - a_0^T x + b_0\right) \ : \ \begin{array}{c} (Q + \mu I)x = -\frac{1}{2}c \\ \|x\| = 1 \end{array} \right\}, \tag{2}$$

which is polynomial-time solvable because, after restricting to the affine subspace $(Q + \mu I)x = -\frac{1}{2}c$, it is an instance of the equality-constrained trust-region subproblem. This optimization can be interpreted as examining the optimal solutions of (T_0) corresponding to μ, while simultaneously trying to make x feasible with respect to the constraints $-1 \le a_0^T x - b_0 \le 1$. In particular, if the optimal value of (2) is nonnegative, then the associated optimal solution has met both constraints and is thus feasible for (1). This uses the fact that the linear constraints are parallel and cannot both be violated simultaneously. After considering all three μ values, the candidates can be judged one-by-one for feasibility and global optimality.

If the constraints $-1 \le a_0^T x - b_0 \le 1$ are replaced by the more general non-intersecting $a_1^T x \le b_1$ and $a_2^T x \le b_2$, Ye and Zhang's analysis carries through. The first case of binding $(b_1 - a_1^T x)(b_2 - a_2^T x) = 0$ works the same by solving (R_2^c). For the second case, one analogously solves

$$\max_x \left\{ (b_1 - a_1^T x)(b_2 - a_2^T x) \ : \ \begin{array}{c} (Q + \mu I)x = -\frac{1}{2}c \\ \|x\| = 1 \end{array} \right\}. \tag{3}$$

Here again, a nonnegative optimal value guarantees that the associated associated optimal solution $\bar{x}$ satisfies both $a_i^T \bar{x} \le b_i$ $(i = 1, 2)$, which can be argued as follows. We need only eliminate the possibility that, without loss of generality, $a_1^T \bar{x} > b_1$.

We consider two subcases. First, assume for contradiction that $a_1^T \bar{x} > b_1$ and $a_2^T \bar{x} = b_2$, and let $\hat{x}$ be feasible for (T_2) with $a_1^T \hat{x} < b_1$ and $a_2^T \hat{x} = b_2$. Such a point exists when both linear constraints are non-redundant and, as assumed, non-intersecting. Then some convex combination y of $\bar{x}$ and $\hat{x}$ has $a_1^T y = b_1$ and $a_2^T y = b_2$, which contradicts the non-intersecting property. Now assume for contradiction that $a_1^T \bar{x} > b_1$ and $a_2^T \bar{x} > b_2$. Let $\hat{x}$ be any feasible point of (T_2). Then some convex combination y of $\bar{x}$ and $\hat{x}$ has $a_1^T y > b_1$ and $a_2^T y = b_2$ (or possibly the indices 1 and 2 are switched). So we are back to the previous subcase, providing the desired contradiction.

3 Preliminaries

With the result of Sect. 2 as motivation, we now discuss some preliminary items in preparation for Sect. 4. In terms of notation, $\mathcal{F}(T_m)$ and $\mathcal{F}(R_m)$ denote the feasible sets of (T_m) and (R_m), respectively. In addition, given (x, X), we define

$$Y(x, X) := \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}$$

and remark that $X = xx^T$ if and only if $Y(x, X)$ is rank-1. Accordingly, we also say that (x, X) is rank-1 if $X = xx^T$.

We assume throughout that $\mathcal{F}(T_m) \neq \emptyset$ and formally state our assumption that no two hyperplanes defined by $a_i^T x = b_i$ and $a_j^T x = b_j$ intersect within the unit ball.

Assumption 1 For all $i < j$, there exists no $x \in \mathcal{F}(T_m)$ such that $a_i^T x = b_i$ and $a_j^T x = b_j$.

Note that Assumption 1 can be verified in polynomial time by checking a polynomial number of convex feasibility problems. We will also need the following result, which has been discussed in the introduction:

Lemma 1 *Every extreme point* $(x, X) \in \mathcal{F}(R_0)$ *satisfies* $X = xx^T$.

Another important result for Sect. 4 demonstrates how any $(x, X) \in \mathcal{F}(R_m)$ with $a_i^T x < b_i$ gives rise to a special vector $z_i \in \mathcal{F}(T_m)$.

Lemma 2 *Suppose* $(x, X) \in \mathcal{F}(R_m)$ *with* $a_i^T x < b_i$ *for some* i. *Define*

$$z_i := (b_i - a_i^T x)^{-1}(b_i x - X a_i) \tag{4}$$

Then $z_i \in \mathcal{F}(T_m)$.

Proof First, the ith SOC-RLT constraint of (R_m) guarantees $\|z_i\| \leq 1$. Furthermore, $X \succeq xx^T$ implies

$$\begin{aligned}(b_i - a_i^T x)(b_i - a_i^T z_i) &= (b_i - a_i^T x)b_i - a_i^T(b_i x - X a_i) = b_i^2 - 2b_i a_i^T x + a_i^T X a_i \\ &\geq b_i^2 - 2b_i a_i^T x + a_i^T xx^T a_i = (b_i - a_i^T x)^2 > 0.\end{aligned}$$

Finally, for all $j \neq i$, the RLT constraints of (R_m) and symmetry of X imply

$$\begin{aligned}(b_i - a_i^T x)(b_j - a_j^T z_i) &= (b_i - a_i^T x)b_j - a_j^T(b_i x - X a_i) \\ &= b_i b_j - b_j a_i^T x - b_i a_j^T x + a_i^T X a_j \\ &\geq 0.\end{aligned}$$

This completes the proof. □

Finally, Sect. 4 requires a simple fact about the extreme points of the intersection of a compact convex set with a half-space.

Lemma 3 *Let* C *be a compact convex set, and let* H *be a half-space. Every extreme point of* $C \cap H$ *may be expressed as the convex combination of at most two extreme points in* C.

Proof Suppose that H is defined by the linear inequality $\alpha^T x \leq \beta$, and let $\bar{x} \in C \cap H$ be extreme. Since $\bar{x} \in C$, we may write $\bar{x} = \sum_{k \in K} \bar{\lambda}_k \bar{x}_k$, where K is some index set, each $\bar{x}_k$ is extreme in C, each $\bar{\lambda}_k > 0$, and $\sum_{k \in K} \bar{\lambda}_k = 1$.

For a vector variable λ with the same length as $\bar{\lambda}$, define the linear function $x(\lambda) := \sum_{k\in K} \lambda_k \bar{x}_k$. In words, $x(\lambda)$ outputs the linear combination $\sum_{k\in K} \lambda_k \bar{x}_k$ of the fixed $\bar{x}_k$. For example, $x(\bar{\lambda}) = \bar{x}$. Also define the polytope $L := \{\lambda \geq 0 : \alpha^T x(\lambda) \leq \beta, \sum_{k\in K} \lambda_k = 1\}$, which has two features. First, because $\lambda \geq 0$ and the sum of its entries equals 1, $x(\lambda)$ defines a convex combination of the vectors $\bar{x}_k$; so $x(\lambda) \in C$. Second, the constraint $\alpha^T x(\lambda) \leq \beta$ ensures $x(\lambda) \in H$. Overall, $\lambda \in L$ implies $x(\lambda) \in C \cap H$. In addition, after adding a slack variable, one can recast L as a standard-form polytope with the general structure $\{z \geq 0 : Az = b\}$, where the number of rows in A is 2, i.e., the basis size is 2. Then every extreme point z has at most two positive entries, which also ensures that every extreme point $\lambda \in L$ has at most two positive entries.

It holds that $\bar{\lambda} > 0$ is feasible for L since $\bar{x} = x(\bar{\lambda}) \in C \cap H$. Hence, we can write $\bar{\lambda} = \sum_j \rho_j \lambda^j$, where $\sum_j \rho_j = 1$, each $\rho_j > 0$, and each λ^j is extreme in L. By expanding, this means $\bar{x} = \sum_j \rho_j x(\lambda^j)$ with each $x(\lambda^j) \in C \cap H$. Since $\bar{x}$ is extreme in $C \cap H$, it holds that every $x(\lambda^j) = \bar{x}$. Since λ^j has at most two positive entries, this completes the proof. □

4 The result

We would like to prove $v(R_m) = v(T_m)$ under Assumption 1, and we will accomplish this by showing that every extreme point (x, X) of the compact $\mathcal{F}(R_m)$ satisfies $X = xx^T$. Our proof is by induction on m, where Lemma 1 with $m = 0$ serves as the base case, i.e., every extreme $(x, X) \in \mathcal{F}(R_0)$ satisfies $X = xx^T$. The induction hypothesis is thus as follows:

Assumption 2 Given $1 \leq i \leq m$, every extreme point $(x, X) \in \mathcal{F}(R_{i-1})$ satisfies $X = xx^T$.

Furthermore, our proof is broken down into two cases:

Case 1 *Some constraint $a_i^T x \leq b_i$ is redundant for $\mathcal{F}(T_m)$.*

Case 2 *No constraint $a_i^T x \leq b_i$ is redundant for $\mathcal{F}(T_m)$.*

Case 1 can be handled immediately.

Theorem 1 *For Case 1, every extreme $(x, X) \in \mathcal{F}(R_m)$ satisfies $X = xx^T$.*

Proof Without loss of generality, assume $a_m^T x \leq b_m$ is redundant. Because $\mathcal{F}(R_{m-1})$ is a relaxation of $\mathcal{F}(R_m)$, $(x, X) \in \mathcal{F}(R_{m-1})$. Hence, by Assumption 2, $Y(x, X)$ may be expressed as the convex combination of rank-1 matrices of the form $\binom{1}{\bar{x}}\binom{1}{\bar{x}}^T$, where each $\bar{x} \in \mathcal{F}(T_{m-1})$. Such $\bar{x}$ also satisfy $a_m^T \bar{x} \leq b_m$ by redundancy, and so (x, X) is the convex combination of points $(\bar{x}, \bar{x}\bar{x}^T) \in \mathcal{F}(R_m)$. Since (x, X) is extreme, this implies $X = xx^T$. □

Now assume Case 2. Propositions 1 and 2 are the key results leading to Theorem 2 below, but first we state a critical lemma that applies in Case 2.

Lemma 4 *For Case 2, suppose x satisfies $\|x\| \le 1$ and $a_i^T x = b_i$ for some i. Then $x \in \mathcal{F}(T_m)$.*

Proof Because $a_i^T x \le b_i$ is non-redundant and $\mathcal{F}(T_m)$ is convex, there exists $\bar{x} \in \mathcal{F}(T_m)$ such that $a_i^T \bar{x} = b_i$. Now suppose $x \notin \mathcal{F}(T_m)$, i.e., $a_j^T x > b_j$ for all $j \in \mathcal{J}$, where $\mathcal{J} \neq \emptyset$ is some collection of indices not including i. Then some convex combination of $\bar{x}$ and x, say $\hat{x}$, is feasible for (T_m) and satisfies $a_i^T \hat{x} = b_i$ and $a_j^T \hat{x} = b_j$ for some $j \in \mathcal{J}$. However, this contradicts Assumption 1, so x is in fact feasible for (T_m). $\square$

Proposition 1 *For Case 2, let $(x, X) \in \mathcal{F}(R_m)$ be extreme such that some SOC-RLT constraint is active. Then $X = xx^T$.*

Proof Assume without loss of generality that $\|b_1 x - X a_1\| = b_1 - a_1^T x$, and consider (R_1) based on the single constraint $a_1^T x \le b_1$. Since (x, X) is also in $\mathcal{F}(R_1)$, Assumption 2 implies

$$Y := Y(x, X) = \sum_k \lambda_k \binom{1}{x_k}\binom{1}{x_k}^T,$$

where $\sum_k \lambda_k = 1$ and, for each k, $x_k \in \mathcal{F}(T_1)$ and $\lambda_k > 0$. Note that

$$\begin{aligned}\begin{pmatrix} b_1 - a_1^T x \\ b_1 x - X a_1 \end{pmatrix} = Y \begin{pmatrix} b_1 \\ -a_1 \end{pmatrix} &= \sum_k \lambda_k \begin{pmatrix} 1 \\ x_k \end{pmatrix}\begin{pmatrix} 1 \\ x_k \end{pmatrix}^T \begin{pmatrix} b_1 \\ -a_1 \end{pmatrix} \\ &= \sum_k \lambda_k (b_1 - a_1^T x_k) \begin{pmatrix} 1 \\ x_k \end{pmatrix}. \end{aligned} \tag{5}$$

Since the left-hand side of (5) is on the boundary of the SOC, each summand $\lambda_k(b_1 - a_1^T x_k)\binom{1}{x_k}$ on the right is either 0 or parallel to $\binom{b_1 - a_1^T x}{b_1 x - X a_1}$ (which itself could be 0). As $\lambda_k > 0$ and $\binom{1}{x_k} \neq 0$, we can thus separate the indices k into two groups (using new indices j and ℓ to distinguish the groups):

$$Y = \sum_{j : a_1^T x_j = b_1} \lambda_j \binom{1}{x_j}\binom{1}{x_j}^T + \sum_{\ell : a_1^T x_\ell < b_1} \lambda_\ell \binom{1}{x_\ell}\binom{1}{x_\ell}^T.$$

Each x_ℓ, if there exist any, must be equal to $z_1 := (b_1 - a_1^T x)^{-1}(b_1 x - X a_1)$ since $\binom{1}{x_\ell}$ is parallel to $\binom{b_1 - a_1^T x}{b_1 x - X a_1}$. Note that the existence of at least one ℓ implies $a_1^T x < b_1$ so that z_1 is well-defined. By (4) and Lemma 2, $z_1 \in \mathcal{F}(T_m)$. In addition, each $x_j \in \mathcal{F}(T_m)$ by Lemma 4. Overall, we see that Y is the convex combination of rank-1 solutions to (T_m). Therefore, Y is rank-1 because (x, X) is extreme. $\square$

Proposition 2 *For Case 2, let $(x, X) \in \mathcal{F}(R_m)$ be extreme such that no SOC-RLT constraint is active. Then $X = xx^T$.*

Proof Suppose that all RLT constraints corresponding to the pairs $(1, m), \ldots, (m-1, m)$ are inactive at (x, X). Then (x, X) is also extreme for (R_{m-1}), and so $X = xx^T$ by Assumption 2. On the other hand, suppose the (i, m)th RLT constraint is tight; say $i = 1$ without loss of generality. We will derive a contradiction that (x, X) is extreme to complete the proof.

We first claim that the other RLT constraints corresponding to $(2, m), \ldots, (m-1, m)$ are inactive. Let z_m be given by (4); it is feasible for (T_m) by Lemma 2. Moreover, the proof of Lemma 2 shows that $a_k^T z_m = b_k$ if and only if the (k, m)th RLT constraint is tight. Since at most one $a_k^T z_m \le b_k$ can be tight by Assumption 1, at most one of the (k, m)th RLT constraints can be active, as claimed.

Let G denote the intersection of $\mathcal{F}(R_{m-1})$ with the single RLT constraint $b_1 b_m - b_m a_1^T x - b_1 a_m^T x + a_1^T X a_m \ge 0$. So (x, X) is extreme for G. Then Lemma 3 implies (x, X) can be expressed as a convex combination of at most two extreme points of (R_{m-1}). By Assumption 2, this means $\operatorname{rank}(Y) \le 2$, where $Y := Y(x, X)$.

Defining $s := \binom{b_1}{-a_1}$ and $t := \binom{b_m}{-a_m}$, we see

$$\begin{aligned} s^T Y s &= \begin{pmatrix} b_1 \\ -a_1 \end{pmatrix}^T \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \begin{pmatrix} b_1 \\ -a_1 \end{pmatrix} = b_1^2 - 2b_1 a_1^T x + a_1^T X a_1 \\ &\ge b_1^2 - 2b_1 a_1^T x + a_1^T x x^T a_1 = (b_1 - a_1^T x)^2 \\ &> 0 \end{aligned}$$

and similarly $t^T Y t > 0$. Notice also that the tight RLT constraint can be expressed as $s^T Y t = 0$. Next consider the equation

$$W := \begin{pmatrix} s^T \\ t^T \\ I \end{pmatrix} Y \begin{pmatrix} s & t & I \end{pmatrix} = \begin{pmatrix} s^T Y s & s^T Y t & s^T Y \\ t^T Y s & t^T Y t & t^T Y \\ Y s & Y t & Y \end{pmatrix} = \begin{pmatrix} s^T Y s & 0 & s^T Y \\ 0 & t^T Y t & t^T Y \\ Y s & Y t & Y \end{pmatrix}.$$

We have $W \succeq 0$ and $\operatorname{rank}(W) \le \operatorname{rank}(Y) \le 2$. Then the Schur complement theorem implies $M := Y - (s^T Y s)^{-1}(Y s s^T Y) - (t^T Y t)^{-1}(Y t t^T Y) \succeq 0$ and $\operatorname{rank}(M) = \operatorname{rank}(W) - 2 \le 2 - 2 = 0$, i.e., $M = 0$ or

$$Y = (s^T Y s)^{-1}(Y s)(Y s)^T + (t^T Y t)^{-1}(Y t)(Y t)^T.$$

We now prove several properties of

$$Y s = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \begin{pmatrix} b_1 \\ -a_1 \end{pmatrix} = \begin{pmatrix} b_1 - a_1^T x \\ b_1 x - X a_1 \end{pmatrix}.$$

First, $Y s$ lies in the interior of the SOC because $\|b_1 x - X a_1\| < b_1 - a_1^T x$. In particular, z_1 in (4) is well-defined with $\|z_1\| < 1$. Furthermore, $t^T Y s = 0$ implies $t^T \binom{1}{z_1} = 0$, or equivalently $a_m^T z_1 = b_m$. Hence, Lemma 4 implies $z_1 \in \mathcal{F}(T_m)$. In a similar manner, we can prove from $Y t$ that z_m defined by (4) is feasible for (T_m) with $a_1^T z_m = b_1$. Also, we see $z_1 \ne z_m$ by Assumption 1.

Summarizing, $Y = \alpha\binom{1}{z_1}\binom{1}{z_1}^T + \beta\binom{1}{z_m}\binom{1}{z_m}^T$ for appropriate positive scalars α, β and $z_1, z_m \in \mathcal{F}(T_m)$ with $z_1 \neq z_m$. Since the top-left entry in Y equals 1, this is a proper convex combination of points in $\mathcal{F}(R_m)$. However, this contradicts that (x, X) is extreme. $\square$

We are now ready to state the desired theorem regarding Case 2 and the main result of the paper (Corollary 1).

Theorem 2 *For Case 2, every extreme* $(x, X) \in \mathcal{F}(R_m)$ *satisfies* $X = xx^T$.

Proof Proposition 1 covers the case when (x, X) has an active SOC-RLT constraint, while Proposition 2 handles when (x, X) has no such active constraint. $\square$

Corollary 1 *Under Assumption 1,* $v(R_m) = v(T_m)$.

Proof Theorem 1 covers Case 1, and Theorem 2 covers Case 2. $\square$

In [2], Burer and Anstreicher gave a counter-example for which $v(R_2) < v(T_2)$ when Assumption 1 is violated:

$$Q = \begin{pmatrix} 2 & 3 & 12 \\ 3 & -19 & 6 \\ 12 & 6 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 14 \\ 14 \\ 9 \end{pmatrix}, \quad -x_1 \le \tfrac{1}{2}, \quad x_1 + \tfrac{6}{5}x_2 \le 0.$$

In this instance, $v(T_2) \approx -12.9419$ with $x^* \approx (-0.8536, 0.2947, 0.4294)^T$, while $v(R_2) \approx -13.8410$ with optimal

$$\bar{x} \approx \begin{pmatrix} -0.3552 \\ 0.3881 \\ -0.2119 \end{pmatrix}, \quad \bar{X} \approx \begin{pmatrix} 0.2595 & -0.2248 & -0.0913 \\ -0.2248 & 0.4495 & -0.0694 \\ -0.0913 & -0.0694 & 0.2911 \end{pmatrix},$$

and the numerical rank of $\bar{X}$ is 3. We wish to examine this counter-example from the viewpoint of our proof. One can verify that $Y(\bar{x}, \bar{X})$ makes both SOC-RLT constraints active in (R_2), and so considering that $(\bar{x}, \bar{X})$ is likely to be extreme in (R_m) from the numerical point of view, Proposition 1 is violated in this case. Of course, Proposition 1 is based on Lemma 4, which heavily uses Assumption 1.

5 An extension

Consider the following assumption, which is slightly relaxed compared to Assumption 1.

Assumption 3 For all $i < j$, there exists no x $\in \mathcal{F}(T_m)$ such that $\|x\| < 1$, $a_i^T x = b_i$, and $a_j^T x = b_j$.

In comparison to Assumption 1, Assumption 3 allows the linear constraints to intersect on the boundary of the unit ball. Every such intersection point on the boundary must clearly be an extreme point of the polyhedron $P := \{x : a_i^T x \le b_i \ (i = 1, \ldots, m)\}$. For example, when the dimension of x is 2, Assumption 3 allows (T_m) to model polytopes P inscribed in the unit disk.

We have the following extension of Corollary 1.

Proposition 3 *Under Assumption 3,* $v(R_m) = v(T_m)$.

Proof For $\epsilon > 0$ small, tighten the constraint $\|x\| \le 1$ of (T_m) to $\|x\| \le 1 - \epsilon$ in order to form a new extended trust region subproblem (eTRS) (T_m^ϵ) satisfying Assumption 1. Relative to (R_m), a suitably modified convex relaxation (R_m^ϵ) can be derived such that $v(R_m^\epsilon) = v(T_m^\epsilon)$ by Corollary 1. The result follows because $v(R_m) = \lim_{\epsilon \to 0} v(R_m^\epsilon) = \lim_{\epsilon \to 0} v(T_m^\epsilon) = v(T_m)$ as all involved feasible sets are compact. $\square$

We end with an application of Proposition 3. Consider $\min\{x^T Qx + c^T x : x \in P\}$, where P is the regular pentagon inscribed in the unit disk with $(1, 0)$ as an extreme point and

$$Q = \begin{pmatrix} -8 & 8 \\ 8 & -14 \end{pmatrix}, \qquad c = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

For example, P could be represented by the system

$$\begin{pmatrix} 1.3764 & 1 \\ -0.3249 & 1 \\ -1.0000 & 0 \\ -0.3249 & -1 \\ 1.3764 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} 1.3764 \\ 0.8507 \\ 0.8090 \\ 0.8507 \\ 1.3764 \end{pmatrix}.$$

Since the redundant constraint $\|x\| \le 1$ is not given explicitly, a reasonable approach would be to solve the following relaxation that only contains the RLT constraints:

$$\begin{aligned} \min_{x,X} \quad & Q \bullet X + c^T x \\ \text{s.t. } & X \succeq xx^T \\ & b_i b_j - b_j a_i^T x - b_i a_j^T x + a_i^T X a_j \ge 0 \qquad i < j. \end{aligned}$$

This yields optimal

$$\bar{x} \approx \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \bar{X} \approx \begin{pmatrix} 0.8090 & 0 \\ 0 & 0.8090 \end{pmatrix}, \qquad Q \bullet \bar{X} + c^T \bar{x} \approx -17.7984.$$

Note that the numerical rank of $\bar{X}$ is 2. According to Proposition 3, however, we can obtain the exact optimal value by solving (R_m). Doing so yields

$$x^* \approx \begin{pmatrix} 0.3090 \\ -0.9511 \end{pmatrix}, \qquad X^* \approx \begin{pmatrix} 0.0955 & -0.2939 \\ -0.2939 & 0.9045 \end{pmatrix},$$
$$Q \bullet X^* + c^T x^* \approx -17.4873.$$

Indeed, the numerical rank of X^* is 1, showing that x^* is a global minimizer of $x^T Qx + c^T x$ over P.

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