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A TWO-VARIABLE APPROACH TO THE TWO-TRUST-REGION SUBPROBLEM*

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Abstract. The trust-region subproblem minimizes a general quadratic function over an ellipsoid and can be solved in polynomial time using a semidefinite-programming (SDP) relaxation. Intersecting the feasible set with a second ellipsoid results in the two-trust-region subproblem (TTRS). Even though TTRS can also be solved in polynomial time, existing algorithms do not use SDP. In this paper, we investigate the use of SDP for TTRS. Starting from the basic SDP relaxation of TTRS, which admits a gap, recent research has tightened the basic relaxation using valid second-order-cone inequalities. Even still, closing the gap requires more. For the special case of TTRS in dimension $n = 2$, we fully characterize the remaining valid inequalities, which can be viewed as strengthened versions of the second-order-cone inequalities just mentioned. We also demonstrate that these valid inequalities can be used computationally even when $n > 2$ to solve TTRS instances that were previously unsolved using SDP-based techniques.

Key words. trust-region subproblem, semidefinite programming, nonconvex quadratic programming

AMS subject classifications. 90C20, 90C22, 90C25, 90C26, 90C30

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1. Introduction.

This paper studies the two-trust-region subproblem, called TTRS, which is the minimization of a general quadratic function over the intersection of two full-dimensional ellipsoids:

$$(\text{TTRS}) \qquad v^* := \min_{x\in\mathbb{R}^2} \; x^T Cx + 2\,c^T x \quad \text{s.t.} \;\; (x-a_i)^T A_i (x-a_i) \le 1, \qquad i = 1, 2.$$

The data are $n \times n$ symmetric matrices C, A_i and vectors $c, a_i \in \mathbb{R}^n$. Moreover, each A_i is positive definite so that each set $E_i := \{x : (x-a_i)^T A_i(x-a_i) \le 1\}$ is a full-dimensional ellipsoid with center a_i. Let $F := E_1 \cap E_2$ denote the feasible region of (TTRS). If C is positive semidefinite, then TTRS is solvable in polynomial time using second-order cone programming. So we assume that C is not positive semidefinite. TTRS was originally introduced by Celis, Dennis, and Tapia [10] and hence is sometimes also called the *CDT problem*.

TTRS is a generalization of the classical *trust-region subproblem*, called TRS, that minimizes $x^T Cx + 2\,c^T x$ over a single ellipsoid $(x - a_1)^T A_1(x - a_1) \le 1$, which can be assumed to be the unit ball $\|x\| \le 1$ without loss of generality. TRS serves as the basis of trust-region methods for nonlinear optimization [11] and, even though nonconvex, can be solved efficiently in theory and practice [13, 17, 19]. In particular, the papers [12, 23] consider the polynomial-time complexity of constructing an ϵ-optimal solution of TRS. Moreover, the optimal value of TRS equals the optimal value of the following polynomial-time solvable semidefinite program (SDP) [19]

$$(1) \qquad \min\left\{ C \bullet X + 2\,c^T x \;:\; A_1 \bullet X - 2\,a_1^T A_1 x + a_1^T A_1 a_1 \le 1,\; \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0 \right\},$$

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where the notation $M \bullet X := \text{trace}(M^T X)$ denotes the inner product of symmetric matrices. For additional background on TRS, we refer the reader to [19, section 1.1].

In addition to TTRS, several other generalizations of TRS have been studied; see [19] for an early discussion. For example, [22] investigates the addition of a single linear inequality to TRS, which is shown to have an exact convex relaxation gotten by adding a second-order-cone constraint to (1). The paper [24] discusses an extension having two parallel, nonintersecting linear constraints, which can be solved in polynomial time by subdividing the problem into cases. Extending [24], the papers [8, 9] show that adding any number of linear constraints to TRS—as long as the constraints do not intersect in the ellipsoid—can be solved in polynomial time by adding extra linear and second-order-cone constraints to (1). Two recent papers have studied the intersection of TRS with a general polyhedron P. First, [15] provides a sufficient condition on the data of the problem, including P, under which a simple extension of (1) is tight. Second, [6] shows that the problem with P can be solved in polynomial time as long as the number of faces of P within the ellipsoid is polynomial. The approach of [6] is combinatorial in nature by subdividing the problem into cases and, in particular, does not make use of convex relaxation.

TTRS has itself received considerable attention. Optimality conditions are studied in [18], which also discusses much of the related literature from the 1990s. A recent paper looking at global and local optimality conditions through the lens of copositivity is [7]. The papers [1, 4] study conditions when the basic SDP relaxation of TTRS, i.e., the relaxation that adds the second constraint $A_2 \bullet X - 2\, a_2^T A_2 x + a_2^T A_2 a_2 \le 1$ to (1), is tight. On the other hand, [24] develops a trajectory-following procedure that solves TTRS generally. This procedure for TTRS is not proved to be polynomial but appears to be practically quite efficient. The authors of [24] also questioned whether there might exist exact polynomial-time formulations of TTRS. The paper [8] shed some light on this question by providing valid second-order-cone constraints tightening the basic SDP relaxation of TTRS, but even these relaxations are still not tight. In section 2, we review the known relaxations of TTRS that are based on the variables (x, X).

Most recently, [5] has demonstrated that TTRS can be solved in polynomialtime using an algorithm of [3] to determine whether two or more quadratic forms share a common zero; see also [14]. While this establishes the polynomial complexity of TTRS, we note that the algorithm employed in [5] does not use convex relaxation (similarly to [6] above). We believe it is still interesting to seek the convex relaxation that solves TTRS exactly. For example, the convex relaxation could provide insight into problems for which TTRS appears as a substructure, even when the algorithm of [5] may not be applicable. We are encouraged by the results of [5], which indicate that the goal of finding the tight convex relaxation is a reasonable one.

Hence, in this paper, our goal is to investigate which additional valid constraints in (x, X) are necessary to calculate v^* exactly. Our main theoretical contribution is a full specification of the additional constraints needed when $n = 2$. Section 2 and Table 1 therein provide the description of the precise inequalities required, which are strengthened (or "lifted") versions of the inequalities previously described in [8]. Sections 3 and 4 contain the technical details and proofs. The tools that we develop help us classify the local and global behavior of all quadratics over the intersection of two ellipsoids in $\mathbb{R}^2$.

While our theoretical result is limited in dimension, we believe our results provide significant insight into the nature of TTRS and its solution by convex relaxation methods. Indeed, in section 5, we show how to separate the type of valid inequalities discovered in this paper for general n, which allows us to solve computationally instances of TTRS that were previously unsolved using SDP-based techniques.

1.1. Notation, terminology, and a simplifying assumption. We use standard notation for Euclidean spaces of vectors and matrices. The set $\mathbb{R}^n$ denotes column vectors of size n, and $\mathcal{S}^n$ denotes all symmetric $n \times n$ matrices. The inner product of $M, N \in \mathcal{S}^n$ is $M \bullet N := \text{trace}(M^T N)$. The subset $\mathcal{S}^n_+ \subseteq \mathcal{S}^n$ consists of positive semidefinite matrices, that is, symmetric matrices with all nonnegative eigenvalues. The notation Closure$(\cdot)$ denotes the closure operation, and Conv$(\cdot)$ denotes the convex hull. For a convex cone $\mathcal{K}$, Ext$(\mathcal{K})$ is the set of extreme rays of $\mathcal{K}$.

In this paper, we will deal with quadratic functions of the general form $f(x) = x^T R x + 2\, r^T x + \rho$. Without loss of generality, R is symmetric. However, we will sometimes write f using a nonsymmetric representation, e.g., $f(x) = (\beta - \alpha^T x)(\delta - \gamma^T x) = \beta\delta - \beta\gamma^T x - \delta\alpha^T x + x^T \alpha\gamma^T x$. This is simply for convenience and does not change the fact that the Hessian of the quadratic is assumed symmetric. Given f, we will also sometimes find it convenient to refer to the Hessian of f without specifying R. In these cases, we simply write Hess(f).

The geometry of the feasible region F will play a significant role in our analysis, and so we define several sets describing different portions of $F = E_1 \cap E_2$. We assume that F is full dimensional in $\mathbb{R}^n$ and that $E_1 \subsetneq E_2$ and $E_2 \subsetneq E_1$. In particular, we do not consider cases with F being empty, a singleton, or equal to E_1 or E_2. Let $\text{int}(E_i) := \{x \in E_i : (x - a_i)^T A_i (x - a_i) < 1\}$ be the interior of E_i and $\text{bd}(E_i) := \{x \in E_i : (x - a_i)^T A_i (x - a_i) = 1\}$ be its boundary for $i = 1, 2$. We also use int(F) to denote the nonempty interior of F, and bd(F) to denote the boundary. Finally, define $\text{vert}(F) := \text{bd}(E_1) \cap \text{bd}(E_2)$ to be the points on the boundaries of both ellipsoids; these are the *vertices*.

For the sake of simplicity, throughout this paper we assume that $x \in \text{vert}(F)$ implies $A_1 x - a_1$ and $A_2 x - a_2$ are linearly independent. In other words, the constraint gradients are independent at x. Note that, when $n = 2$ and F is full dimensional, $E_1 \subsetneq E_2$ and $E_2 \subsetneq E_1$ (as assumed above), the set vert(F) is finite with cardinality 1, 2, 3, or 4. This simplifying assumption rules out the case that the cardinality of vert(F) is odd. A slightly more careful presentation is required if the assumption is not satisfied, but the major conclusions of the paper still hold.

2. Relaxations of TTRS. In this section, we review existing SDP relaxations in the variables (x, X) for TTRS and discuss how to tighten the relaxations using quadratic functions that are nonnegative over the feasible region F. The main point is Theorem 4.3, which specifies a subset of valid quadratics that close the SDP relaxation gap completely. Technical details and the proofs are given in sections 3–4.

2.1. Existing relaxations. Because (TTRS) is nonconvex, a reasonable approach is to relax it as a SDP that can be solved in polynomial time [8, 21, 24]:

$$\text{(SDP)} \qquad v(\text{SDP}) := \min \; C \bullet X + 2\, c^T x$$
$$\text{s.t.} \quad A_i \bullet X - 2\, a_i^T A_i x + a_i^T A_i a_i \le 1, \qquad i = 1, 2,$$
$$(x, X) \in \text{PSD},$$

where

$$\text{PSD} := \left\{ (x, X) : \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0 \right\}.$$

We call this the *basic SDP relaxation*. Note that by the Schur complement theorem, $(x, X) \in \text{PSD}$ if and only if $X \succeq xx^T$. Also, when

$$\operatorname{rank} \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = 1,$$

then $X = xx^T$ and $v(\text{SDP}) = v^*$. However, it is well known that the optimal value of (SDP) is in general strictly less than the optimal value of (TTRS), i.e., $v(\text{SDP}) < v^*$.

The paper [8] proposes a method to strengthen (SDP). Let $A_2^{1/2}$ be the positive definite square root of A_2, and rewrite the second ellipsoidal constraint as the second-order-cone constraint $\|A_2^{1/2}(x - a_2)\| \le 1$. In addition, let $\alpha^T x \le \beta$ be any valid inequality that supports the first ellipsoid E_1. Then the following quadratic second-order-cone inequality is valid for F:

$$\begin{aligned} \|A_2^{1/2}(\beta\, x - \alpha^T x \cdot x - \beta\, a_2 + \alpha^T x \cdot a_2)\| &= \|A_2^{1/2}(\beta - \alpha^T x)(x - a_2)\| \\ &= (\beta - \alpha^T x)\|A_2^{1/2}(x - a_2)\| \\ &\le \beta - \alpha^T x. \end{aligned}$$

Moreover, this inequality may be linearized via X and added to the basic SDP relaxation:

$$\|A_2^{1/2}(\beta\, x - X\alpha - \beta\, a_2 + \alpha^T x \cdot a_2)\| \le \beta - \alpha^T x. \tag{2}$$

These inequalities are called SOCRLT (second-order-cone reformulation linearization technique) constraints in [8] since their derivation is closely tied to the regular RLT (reformulation linearization technique [20]) constraints gotten by multiplying two valid linear constraints $\beta - \alpha^T x \ge 0$ and $\delta - \gamma^T x \ge 0$ and then linearizing to get $\beta\delta - \beta \cdot \gamma^T x - \delta \cdot \alpha^T x + \alpha^T X \gamma \ge 0$. Letting SOC denote the set of (x, X) satisfying all possible SOCRLT constraints, [8] proposes to solve

$$\text{(SOC)} \qquad \begin{aligned} v(\text{SOC}) \ := \ \min \ & C \bullet X + 2\, c^T x \\ \text{s.t.} \ & A_i \bullet X - 2\, a_i^T A_i x + a_i^T A_i a_i \le 1, \qquad i = 1, 2, \\ & (x, X) \in \text{PSD} \cap \text{SOC}\,. \end{aligned}$$

Although infinite in number, the SOCRLT constraints can be separated in polynomial time and hence $v(\text{SOC})$ can be efficiently calculated. Nevertheless, [8] shows by example that $v(\text{SDP}) < v(\text{SOC}) < v^*$ in general.

The SOCRLT constraints are defined by multiplying a supporting inequality $\alpha^T x \le \beta$ of E_1 by the second-order-cone representation of E_2. The same relaxation value $v(\text{SOC})$ results if the roles of E_1 and E_2 are switched [8]. Moreover, [8] argues that the entire collection of SOCRLT constraints is equivalent to all possible regular RLT constraints, i.e., those gotten by multiplying a supporting $\beta - \alpha^T x \ge 0$ of E_1 with a supporting $\delta - \gamma^T x \ge 0$ of E_2. In this sense, the SOCRLT constraints are simply a different representation of the RLT constraints.

2.2. Exact relaxations. So the question remains: what additional valid constraints are required beyond those in (SDP) and (SOC) to close the relaxation gap?

Using the approach of [22], we claim that closing the relaxation gap is equivalent to describing the following set, which corresponds to all quadratic functions that are nonnegative over F:

$$\mathcal{K} := \left\{ (R, r, \rho) \in \mathcal{S}^n \times \mathbb{R}^n \times \mathbb{R} : x^T R x + 2 r^T x + \rho \ge 0 \ \forall\ x \in F \right\}.$$

$\mathcal{K}$ is a closed, convex cone [22]. Several classes of elements in $\mathcal{K}$ are readily apparent. For example, $(-A_i, A_i a_i, 1 - a_i^T A_i a_i) \in \mathcal{K}$ corresponds to the ellipsoidal constraint $(x - a_i)^T A_i (x - a_i) \le 1$, and $(\alpha\gamma^T, -\frac{1}{2}(\beta\gamma + \delta\alpha), \beta\gamma) \in \mathcal{K}$ corresponds to the RLT quadratic $(\beta - \alpha^T x)(\delta - \gamma^T x) \ge 0$. Consider the following optimization problem:

$$\begin{aligned} v(\mathcal{K}) \;:=\; \min\;\; & C \bullet X + 2\, c^T x \\ \text{s.t.}\;\; & R \bullet X + 2\, r^T x + \rho \ge 0 \quad \forall\ (R, r, \rho) \in \mathcal{K}. \end{aligned}$$

Proposition 2.1. $v(\mathcal{K}) = v^*$.

Proof. Clearly $v(\mathcal{K}) \le v^*$ because the optimization problem is a relaxation by dropping the constraint $X = xx^T$. In addition, since $(C, c, -v^*) \in \mathcal{K}$ by definition, the constraint $C \bullet X + 2\, c^T x - v^* \ge 0$ guarantees $v(\mathcal{K}) \ge v^*$, as desired. ∎

So closing the gap amounts to enforcing the constraints $R \bullet X + 2\, r^T x + \rho \ge 0$ for all elements in $\mathcal{K}$.

Of course, the definition of $\mathcal{K}$ is quite generic, and it would be helpful to characterize, for example, the extreme rays of $\mathcal{K}$ or a few constraints that capture whole portions of $\mathcal{K}$. Indeed, the two ellipsoidal constraints $A_i \bullet X - 2\, a_i^T A_i x + a_i^T A_i a_i \le 1$ are fundamental, and the SOCRLT constraints capture all the RLT constraints as discussed at the end of the previous subsection. In addition, each (x, X) feasible for (SDP) satisfies $R \bullet X + 2\, r^T x + \rho \ge 0$ for all $(R, r, \rho) \in \mathcal{K}$ with $R \succeq 0$. This can be seen in two steps. First, each feasible (x, X) implies $x \in F$ since

$$(x - a_i)^T A_i (x - a_i) = A_i \bullet xx^T - 2\, a_i^T A_i x + a_i^T A_i a_i \le A_i \bullet X - 2\, a_i^T A_i x + a_i^T A_i a_i \le 1,$$

where the second inequality follows because A_i is positive definite and $(x, X) \in \text{PSD}$. Second, because $R \succeq 0$, $(x, X) \in \text{PSD}$, and $x \in F$,

$$R \bullet X + 2\, r^T x + \rho \ge R \bullet xx^T + 2\, r^T x + \rho \ge 0.$$

To fully characterize elements in $\mathcal{K}$, we would like to define and investigate a proper subcone $\mathcal{G}$ of $\mathcal{K}$ that is guaranteed to contain $\text{Ext}(\mathcal{K})$. In this way, $\mathcal{G}$ gives rise to an SDP relaxation whose optimal value equals v^*:

$$\begin{aligned} v^* = v(\mathcal{G}) \;:=\; \min\;\; & C \bullet X + 2\, c^T x \\ \text{s.t.}\;\; & R \bullet X + 2\, r^T x + \rho \ge 0 \quad \forall\ (R, r, \rho) \in \mathcal{G}. \end{aligned}$$

The main technical approach is to show that $\mathcal{G}$ has a special property as described in Proposition 2.2 below.

We first introduce some definitions. For (R, r, ρ), we write $f = (R, r, \rho)$ and define the function $f(x) := x^T R x + 2 r^T x + \rho$ acting on the vector variable $x \in \mathbb{R}^n$. When $f \in \mathcal{K}$, we say that f is *valid for* F. For $f, g \in \mathcal{K}$, if $f - g$ is also valid, i.e., if $f(x) \ge g(x)$ for all $x \in F$, then we write $f \succeq g$ and say that g *minorizes* f *over* F. We note that minorization is clearly transitive, i.e., $f \succeq g$ and $g \succeq h$ imply $f \succeq h$.

Proposition 2.2. *Let $\mathcal{G} \subseteq \mathcal{K}$ be a cone, not necessarily convex. If, for every $f \in \mathcal{K}$, there exists some $g \in \mathcal{G}$ such that $f \succeq g$, then* $\text{Ext}(\mathcal{K}) \subseteq \mathcal{G}$, *and so* $\text{Closure}(\text{Conv}(\mathcal{G})) = \mathcal{K}$.

Proof. We need to show every $f \in \text{Ext}(\mathcal{K})$ is an element of $\mathcal{G}$. By assumption, we know $f \succeq g$ for some $g \in \mathcal{G}$. If $f \parallel g$, i.e., there exists $\alpha \ge 0$ such that $f = \alpha g$, we are done. On the other hand, when $f \not\parallel g$, the equation $f = (f - g) + g$ shows that f is not extreme in $\mathcal{K}$, a contradiction. The equation $\text{Closure}(\text{Conv}(\mathcal{G})) = \mathcal{K}$ follows because $\mathcal{G}$ contains all the extreme rays of $\mathcal{K}$, which is closed. ∎

Table 1
Our choice of valid quadratic functions generating $\mathcal{G}$.

Valid quadratic	Conditions	Nickname
g_{E_i}	$i \in \{1,2\}$	ellipsoid
f	$\text{Hess}(f) \in \mathcal{S}^2_+$	PSD
$T_{yE_1}T_{yE_2}$	$y \in \text{vert}(F)$	vertex RLT
$T_{yE_1}\,T_{zE_2} + \lambda\,L_{yz}^2$	$y \neq z,\ \ \lambda < 0$ minimal	lifted RLT

From this point forward in sections 2–4, we consider the case when $n = 2$. We will show computationally in section 5 that the insights for dimension $n = 2$ can be used more generally for $n > 2$.

2.3. Choice of valid inequalities. We state our choice of $\mathcal{G}$ here in order to familiarize the reader with some of its features, although the insights that lead to this choice will be developed in sections 3–4. In particular, see Theorem 4.3 in section 4.

Before listing the generators of $\mathcal{G}$, we define several functions. First, define

$$g_{E_i}(x) := 1 - (x - a_i)^T A_i (x - a_i), \qquad i = 1, 2,$$
$$T_{yE_i}(x) := 1 - (y - a_i)^T A_i (x - a_i) \qquad \forall \text{ feasible } y \in \text{bd}(E_i),\ i = 1, 2.$$

In words, $g_{E_i}(x) \geq 0$ defines the ellipsoid E_i. Also, $T_{yE_i}(x) = 0$ defines the tangent line to E_i at y, and the inequality $T_{yE_i}(x) \geq 0$ supports E_i at y. For any two distinct points $y, z \in \mathbb{R}^2$, we also let L_{yz} be a linear function such that $L_{yz}(x) = 0$ defines the unique line passing through y and z. Up to multiplication by a constant, the representation of L_{yz} is unique, and whenever we write L_{yz}, the reader may safely assume that $y \neq z$.

Abusing notation, we also let T_{yE_i} and L_{yz} denote the sets $\{x : T_{yE_i}(x) = 0\}$ and $\{x : L_{yz}(x) = 0\}$, respectively. That is, T_{yE_i} denotes the function defining the tangent line and the tangent line itself, and similarly for L_{yz}.

We require one additional concept. Our choice of $\mathcal{G}$ will contain valid quadratics of the form $f + \lambda g$, where g is itself valid and $\lambda \in \mathbb{R}$. Suppose that both $f + \lambda_1 g$ and $f + \lambda_2 g$ are valid such that $\lambda_1 < \lambda_2$. Then clearly $f + \lambda_1 g \preceq f + \lambda_2 g$, i.e., $f + \lambda_1 g$ minorizes $f + \lambda_2 g$, because the difference $(\lambda_2 - \lambda_1)g$ is valid. It may happen that $f + \lambda_1 g$ is further minorized by $f + \lambda_0 g$ with $\lambda_0 < \lambda_1$. This leads to the concept of λ being *minimal* in $f + \lambda g$, which we establish in the following proposition.

Proposition 2.3. *Let f, g be quadratics with $g \in \mathcal{K}$. Suppose $f + \bar{\lambda} g$ is valid for some $\bar{\lambda} \in \mathbb{R}$. Then there exists $\lambda_{\min} \in \mathbb{R}$ such that $f + \lambda g$ is valid if and only if $\lambda \geq \lambda_{\min}$.*

Proof. For all $x \in F$, define

$$\lambda_{\min}(x) := \begin{cases} -g(x)f(x)^{-1} & \text{if } f(x) \neq 0, \\ -\infty & \text{if } f(x) = 0. \end{cases}$$

That is, for each x separately, $\lambda_{\min}(x)$ measures the smallest value of λ such that $f(x) + \lambda g(x) \geq 0$. This means in particular that $\lambda_{\min}(x) \leq \bar{\lambda}$ since $f + \bar{\lambda} g \in \mathcal{K}$ by assumption. Define $\lambda_{\min} := \sup_{x \in F} \lambda_{\min}(x)$. It is then clear that $f + \lambda g$ is valid if and only if $\lambda \geq \lambda_{\min}$. ∎

The generators for $\mathcal{G}$ are listed in Table 1. There are four classes of generators, each corresponding to a type of valid quadratic function. We give each class a nickname for ease of discussion. The first three classes are already known and have been incorporated into existing relaxations as discussed in section 2.1, while the last class is new in this paper.

The first class consists of the two ellipsoid quadratics $g_{E_i}(x) \geq 0$ for $i \in \{1, 2\}$, and the second class consists of all valid quadratics $f(x) = x^T Rx + 2\, r^T x + \rho \geq 0$ such that $\mathrm{Hess}(f) = 2R$ is positive semidefinite. Together, these ellipsoids and PSD quadratics give rise to the basic SDP relaxation (SDP) as discussed after Proposition 2.1. The third class consists of the RLT quadratics $T_{yE_1}(x)T_{yE_2}(x) \geq 0$, where y is a member of the vertex set $\mathrm{vert}(F)$. Since the cardinality of $\mathrm{vert}(F)$ is at most four when $n = 2$, there are at most four such vertex RLT constraints.

The last class contains quadratics that are derived from valid RLT quadratics of the type $T_{yE_1}(x)T_{zE_2}(x) \geq 0$ with $y \neq z$, i.e., the tangents T_{yE_1} and T_{zE_2} supporting different points on different ellipsoids. However, here the RLT quadratic is minorized by the valid quadratic $T_{yE_1}T_{zE_2} + \lambda\, L_{yz}^2$, where λ is minimal and hence $\lambda \leq 0$. In fact we will prove later that $\lambda < 0$. We call these *lifted RLT quadratics* in analogy with the lifting, or strengthening, of valid inequalities in, for example, the area of linear integer programming. Figure 1 depicts a lifted RLT quadratic. In the left, 2-dimensional picture, we have graphed F and marked $\mathrm{vert}(F)$. Also depicted are the tangent lines T_{yE_1} and T_{zE_2}, as well as the line L_{yz} connecting y and z. In the right, 3-dimensional picture, the value of the lifted RLT quadratic is graphed in the vertical dimension over the the boundary $\mathrm{bd}(F)$ of F. Note that the quadratic attains the value 0 at the points y and z as well as one of the vertices. This shows that the lifted RLT minorizes the regular RLT quadratic, which only attains 0 at y and z.

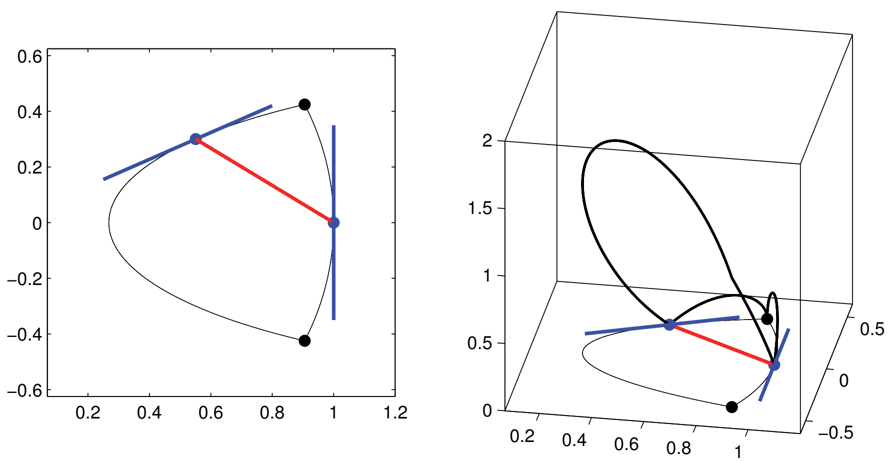

Fig. 1. *A lifted RLT quadratic.*

With the specification of $\mathcal{G}$ given in Table 1, we introduce the following (informally specified) SDP relaxation

$$
(3) \qquad \begin{aligned} v(\mathcal{G}) := \min \quad & C \bullet X + 2\, c^T x \\ \text{s.t.} \quad & A_i \bullet X - 2\, a_i^T A_i x + a_i^T A_i a_i \leq 1, \qquad i = 1, 2, \\ & (x, X) \in \mathrm{PSD}, \\ & (x, X) \text{ satisfies all vertex RLT constraints}, \\ & (x, X) \text{ satisfies all lifted RLT constraints}, \end{aligned}
$$

where the various constraints are gotten by linearizing the valid quadratics. Assuming $\mathcal{G}$ satisfies Proposition 2.2 (as we prove in Theorem 4.3 in section 4), we have the following corollary.

COROLLARY 2.4. *For* $n = 2$, $v(\mathcal{G}) = v^*$.

Proof. As discussed above, the first and second class of quadratics in $\mathcal{G}$ are captured by the constraints $A_i \bullet X - 2\, a_i^T A_i x + a_i^T A_i a_i \le 1$ and $(x, X) \in \mathrm{PSD}$. The third and fourth classes are incorporated directly into the SDP relaxation. ∎

We discuss the practical separation of the lifted RLT constraints (even when $n > 2$) in section 5.

As discussed in section 2.1, the SOCRLT constraints can be viewed as another representation of all RLT constraints. The vertex and lifted RLT constraints in Table 1 imply all RLT constraints, and so the SOCRLT constraints are implied as well. Hence, it is unnecessary to state the SOCRLT constraints in this setting.

3. Local analysis of quadratrics.

We now develop some technical tools that investigate the behavior of general quadratics f on the boundary of F. In this section, we do not assume f is in $\mathcal{K}$, i.e., that f is valid. In section 4, the tools and techniques of this subsection will play a vital role when we fully classify valid f.

3.1. The break concept.

Fix $i \in \{1, 2\}$, and let a feasible $y \in \mathrm{bd}(E_i)$ be given. Consider any locally diffeomorphic parameterization $x(t)$ of $\mathrm{bd}(E_i)$ such that $y = x(t_y)$. That is, x locally takes open intervals in $\mathbb{R}$ to open submanifolds of $\mathrm{bd}(E_i)$ smoothly and invertibly. Given a general quadratic $f(x) = x^T R x + 2\, r^T x + \rho$ and any integer $k \ge 0$, we define $(f \circ x)^{(k)}(t_y)$ to be the kth derivative of the composition $f \circ x$ at the point t_y. By convention, when $k = 0$, the derivative $(f \circ x)^{(0)}(t_y)$ is simply the function value $f(y)$. We also define the *break of* f *at* y *along* $\mathrm{bd}(E_i)$ to be the smallest k such that $(f \circ x)^{(k)}(t_y)$ is nonzero, i.e.,

$$\mathrm{br}(y, \mathrm{bd}(E_i), f) := \min\{k : (f \circ x)^{(k)}(t_y) \ne 0\}.$$

Note that the break equals ∞ if all derivatives are 0, e.g., when $f = g_{E_i}$. In the rest of this section, we state and prove various facts about $\mathrm{br}(y, E_i, f)$. To assist the reader's understanding, specific E_1 and E_2 are used in the facts below—instead of generic E_i and E_j with $i, j \in \{1, 2\}$—but this simplification does not affect the generality of our statements.

First, it is important to note that $\mathrm{br}(y, E_i, f)$ is independent of the locally diffeomorphic parameterization.

PROPOSITION 3.1. *Given feasible* $y \in \mathrm{bd}(E_1)$ *and quadratic* f, *let* $x(t)$ *and* $\hat{x}(s)$ *be any two locally diffeomorphic parameterizations of* $\mathrm{bd}(E_1)$ *such that* $y = x(t_y) = \hat{x}(s_y)$. *Then*

$$\min\{k : (f \circ x)^{(k)}(t_y) \ne 0\} = \min\{k : (f \circ \hat{x})^{(k)}(s_y) \ne 0\}.$$

Equivalently, $\mathrm{br}(y, E_1, f)$ *is independent of the parameterization of* $\mathrm{bd}(E_1)$.

Proof. We may write the parameterizations x and $\hat{x}$ locally as diffeomorphisms

$$x : T \to W, \quad t \mapsto x(t),$$
$$\hat{x} : S \to W, \quad s \mapsto \hat{x}(s),$$

where the open domains $T \ni t_y$ and $S \ni t_s$ are subsets of $\mathbb{R}$ and the range W is a subset of $\mathrm{bd}(E_1)$. Let $\hat{x}^{-1}$ be the (local) inverse of $\hat{x}$. Then

$$(f \circ x)(t) = (f \circ \hat{x} \circ \hat{x}^{-1} \circ x)(t) = (f \circ \hat{x})(\hat{x}^{-1}(x(t))).$$

Letting $g := \hat{x}^{-1} \circ x$, we write $(f \circ x)(t) = (f \circ \hat{x})(g(t))$. By Faà di Bruno's formula, a generalization of the chain rule, at $t = t_y$,

$$(f \circ x)^{(k)}(t_y) = \sum_{i=1}^{k} (f \circ \hat{x})^{(i)}(s_y) \cdot B_{ki}(t_y),$$

where the functions B_{ki} are polynomials in the derivatives of g. This shows that

$$(f \circ \hat{x})^{(i)}(s_y) = 0 \;\forall\; i = 1, \dots, k \quad \Longrightarrow \quad (f \circ x)^{(i)}(t_y) = 0 \;\forall\; i = 1, \dots, k.$$

So $\min\{k : (f \circ x)^{(k)}(t_y) \neq 0\} \geq \min\{k : (f \circ \hat{x})^{(k)}(s_y) \neq 0\}$. By symmetry of x and $\hat{x}$, the reverse inequality must also hold, which completes the proof. $\square$

As a corollary, the break does not change under affine transformation.

COROLLARY 3.2. *Let $\mathcal{A} : \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible, affine transformation. Given feasible $y \in \operatorname{bd}(E_1)$ and quadratic f, define $\hat{y} := \mathcal{A}(y)$, $\hat{E_1} := \mathcal{A}(E_1)$, and $\hat{f} := f \circ \mathcal{A}^{-1}$. Then* $\operatorname{br}(y, E_1, f) = \operatorname{br}(\hat{y}, \hat{E_1}, \hat{f})$.

Proof. Suppose x is a locally diffeomorphic parameterization of $\operatorname{bd}(E_1)$ and $y = x(t_y)$. Then $\hat{x} := \mathcal{A} \circ x$ is a locally diffeomorphic parameterization of $\operatorname{bd}(\hat{E_1})$ and $\hat{y} = \hat{x}(t_y)$. By the preceding proposition, $\operatorname{br}(y, E_1, f)$ and $\operatorname{br}(\hat{y}, \hat{E_1}, \hat{f})$ may be calculated by examining the derivatives of $(f \circ x)(t)$ and $(\hat{f} \circ \hat{x})(t)$ at $t = t_y$, respectively. In addition, for all t in the domain of x, it holds that

$$(\hat{f} \circ \hat{x})(t) = (f \circ \mathcal{A}^{-1} \circ \mathcal{A} \circ x)(t) = (f \circ x)(t),$$

i.e., $\hat{f} \circ \hat{x}$ and $f \circ x$ are the same function. It follows that $\operatorname{br}(\hat{y}, \hat{E_1}, \hat{f}) = \operatorname{br}(y, E_1, f)$. $\square$

The following proposition describes how breaks behave under sums and products of quadratics.

PROPOSITION 3.3. *Given feasible $y \in \operatorname{bd}(E_1)$ and quadratics f and g, define $a := \operatorname{br}(y, E_1, f)$ and $b := \operatorname{br}(y, E_1, g)$. It holds that*

(i) *for any $\lambda, \mu \in \mathbb{R}$, $\operatorname{br}(y, E_1, \lambda f + \mu g) \geq min\{a, b\}$ with equality if $a \neq b$ and $\lambda\mu \neq 0$;*

(ii) *if f and g are linear, then* $\operatorname{br}(y, E_1, fg) = a + b$.

Proof. To prove (i), we define $h := \lambda f + \mu g$ and apply the definition of $\operatorname{br}(y, E_1, h)$. Without loss of generality, assume $\min\{a, b\} = a$. Note that, for all $k \geq 0$, $(h \circ x)^{(k)}(t_y) = \lambda (f \circ x)^{(k)}(t_y) + \mu (g \circ x)^{(k)}(t_y)$. Hence,

$$k < a \quad \Longrightarrow \quad (h \circ x)^{(k)}(t_y) = \lambda \cdot 0 + \mu \cdot 0 = 0.$$

Hence, $\operatorname{br}(y, E_1, h) \geq a$, as claimed. If, in addition, $a < b$ and $\mu \neq 0$, then

$$k = a \quad \Longrightarrow \quad (h \circ x)^{(k)}(t_y) = \lambda \cdot 0 + \mu \cdot (f \circ x)^{(k)}(t_y) \neq 0,$$

which proves $\operatorname{br}(y, E_1, h) = a$.

To prove (ii), we apply the product rule to derive that, for all $k \geq 0$,

$$((fg) \circ x)^{(k)}(t_y) = \sum_{j=0}^{k} \binom{k}{j} \cdot (f \circ x)^{(k-j)}(t_y) \cdot (g \circ x)^{(j)}(t_y).$$

When $k < a + b$, for all $j \leq k$, it must hold that $k - j < a$ or $j < b$. Hence, every term in the above summand is zero by assumption. When $k = a + b$, the only nonzero summand corresponds to $j = b$ and equals $\binom{k}{b} \cdot (f \circ x)^{(a)}(t_y) \cdot (g \circ x)^{(b)}(t_y) \neq 0$. We have thus shown that $\operatorname{br}(y, E_1, fg) = a + b$. $\square$

TABLE 2
Breaks for feasible $y \in \mathrm{bd}(E_1)$, $z \in \mathrm{bd}(E_2)$ such that $y \neq z$ for various quadratics f.

f	$\mathrm{br}(y, E_1, f)$	$\mathrm{br}(y, E_2, f)$ when $y \in \mathrm{vert}(F)$	$\mathrm{br}(z, E_2, f)$
g_{E_1}	∞	≥ 1	(not needed)
$T_{yE_1}^2$	4	2	(not needed)
$T_{yE_1}T_{yE_2}$	3	3	(not needed)
$T_{yE_1}L_{yz}$	3	(not needed)	1
$T_{yE_1}T_{zE_2}$	2	(not needed)	2
L_{yz}^2	2	2	2

The break concept will be significant because, as we will see in what follows, feasible zeros $y \in \mathrm{bd}(E_i)$ of f, which are also local minimizers of f over F, often have relatively high breaks, e.g., $\mathrm{br}(y, E_i, f)$ may equal 2, 3, 4, or even ∞. When this occurs, we can use the zero derivatives indicated by the break as a tool for classifying all quadratic functions possessing that particular break at y. For example, if $y \in \mathrm{bd}(E_1)$ with $\mathrm{br}(y, E_1, f) = 3$ and $y = x(t_y)$, then the three equations $f(y) = (f \circ x)^{(1)}(t_y) = (f \circ x)^{(2)}(t_y) = 0$ give three linear equations that $f = (R, r, \rho)$ must satisfy.

3.2. Breaks for specific quadratics. In section 4, we will require a simple and effective way to check, given two valid quadratics f and g, whether some positive multiple of g minorizes f. As it will turn out, the technique will depend heavily on the zeros and breaks of f and g. Hence, in this subsection we precalculate the breaks for some specific quadratics as reference for later use; see Table 2.

We first look at the breaks of some linear functions, which are the building blocks of most of the quadratics in Table 2.

LEMMA 3.4. *Let feasible $y \in \mathrm{bd}(E_1)$ and z, $w \in \mathbb{R}^2$ with $T_{yE_1}(z) \neq 0$, $L_{zw}(y) \neq 0$ be given. Then $\mathrm{br}(y, E_1, T_{yE_1}) = 2$, $\mathrm{br}(y, E_1, L_{yz}) = 1$, and $\mathrm{br}(y, E_1, L_{zw}) = 0$.*

Proof. By Proposition 3.1 and Corollary 3.2, we may assume without loss of generality that E_1 is the unit ball $\{x : \|x\| \leq 1\}$. Let $x(t) := (\cos t, \sin t)^T$ be the standard parameterization of $\mathrm{bd}(E_1)$, and suppose t_y satisfies $y = x(t_y)$. Also let $T_{yE_1}(x)$ and $L_{yz}(x)$ be represented as $1 - y^T x$ and $l_0 - l^T x$.

We use the specific form of T_{yE_1} to calculate the derivatives $(T_{yE_1} \circ x)^{(k)}(t)$ explicitly:

$$(T_{yE_1} \circ x)^{(0)}(t) = 1 - \cos t_y \cos t - \sin t_y \sin t,$$
$$(T_{yE_1} \circ x)^{(1)}(t) = \cos t_y \sin t - \sin t_y \cos t,$$
$$(T_{yE_1} \circ x)^{(2)}(t) = \cos t_y \cos t + \sin t_y \sin t.$$

Evaluated at t_y, we have $(T_{yE_1} \circ x)^{(0)}(t_y) = (T_{yE_1} \circ x)^{(1)}(t_y) = 0$ and $(T_{yE_1} \circ x)^{(2)}(t_y) = 1 \neq 0$. So $\mathrm{br}(y, E_1, T_{yE_1}) = 2$. For L_{yz},

$$(L_{yz} \circ x)^{(0)}(t) = l_0 - l_1 \cos t - l_2 \sin t,$$
$$(L_{yz} \circ x)^{(1)}(t) = l_1 \sin t - l_2 \cos t.$$

Evaluated at t_y, we have $(L_{yz} \circ x)^{(0)}(t) = l_0 - l_1 y_1 - l_2 y_2 = L_{yz}(y) = 0$. Since $L_{yz} \nparallel T_{yE_1}$, $(L_{yz} \circ x)^{(1)}(t) = l_1 y_2 - l_2 y_1 \neq 0$. So $\mathrm{br}(y, E_1, L_{yz}) = 1$. Finally, as $(L_{zw} \circ x)^{(0)}(t_y) = L_{zw}(y) \neq 0$, $\mathrm{br}(y, E_1, L_{zw}) = 0$. ∎

Intuitively, Lemma 3.4 tells us that a tangent line has break 2 at its support point, while a secant line has break 1. A line that does not pass through y at all has break 0.

Since all but one of the quadratics in Table 2 are products of linear functions, Lemma 3.4 and Proposition 3.3(ii) provide an easy formula to calculate the breaks for those quadratics. For example, suppose $y \in \text{vert}(F)$, $z \in \text{bd}(E_2) \cap F$, and $f = T_{yE_1} L_{yz}$ are given. Then

$$\text{br}(y, E_1, f) = \text{br}(y, E_1, T_{yE_1}) + \text{br}(y, E_1, L_{yz}) = 2 + 1 = 3,$$
$$\text{br}(z, E_2, f) = \text{br}(z, E_2, T_{yE_1}) + \text{br}(z, E_2, L_{yz}) = 0 + 1 = 1.$$

Last, since ellipsoidal constraints are not products of linear functions, we handle their breaks separately in the following lemma.

LEMMA 3.5. *Let feasible $y \in \text{bd}(E_1)$ be given. Then $\text{br}(y, E_1, g_{E_1}) = \infty$. If in addition, $y \in \text{vert}(F)$, then $\text{br}(y, E_2, g_{E_1}) \geq 1$.*

Proof. $\text{br}(y, E_1, g_{E_1})$ is clearly ∞ as g_{E_1} is zero and constant along E_1^1. If $y \in \text{vert}(F)$, then $g_{E_1}(y) = 0$, so $\text{br}(y, E_2, g_{E_1}) \geq 1$. ◻

3.3. Quadratics for specific breaks. While the previous subsection provides the breaks of some specific quadratics, in this subsection we look for quadratics that satisfy specific breaks. As we mentioned in section 3.1, the higher the breaks a quadratic $f = (R, r, \rho)$ has, the more constrained the entries of (R, r, ρ) become. Based on specified breaks, we can often classify the form of f with the help of Table 2. When we ultimately characterize all valid $f \in \mathcal{K}$ in section 4, we will divide the proof into different cases with respect to different breaks. The results contained in this subsection help us deduce the forms of the quadratics in each case.

We assume throughout this subsection that f is defined by $f(x) = x^T R x + 2\, r^T x + \rho$ for $(R, r, \rho) \in \mathcal{S}^2 \times \mathbb{R}^2 \times \mathbb{R}$. We also define the zeros of f in F (or "null" points) by

$$N := N(f) := \{x \in F : f(x) = 0\}.$$

We do not assume that f is valid.

Lemma 3.6 and Propositions 3.7 and 3.8 below consider the case when f has a zero at a vertex with relatively high breaks.

LEMMA 3.6. *Let f and $y \in \text{vert}(F)$ be given. Let $z \in \mathbb{R}^2$ be arbitrary such that $T_{yE_1}(z) \neq 0$ and $T_{yE_2}(z) \neq 0$. If $br(y, E_1, f) \geq 2$ and $br(y, E_2, f) \geq 2$, then there exists $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that*

$$f = \alpha_1 T_{yE_1}^2 + \alpha_2 T_{yE_1} T_{yE_2} + \alpha_3 L_{yz}^2. \tag{4}$$

Proof. The zero equation $f(y) = 0$ implies

$$y_1^2 R_{11} + 2\, y_1 y_2 R_{21} + y_2^2 R_{22} + 2\, y_1 r_1 + 2\, y_2 r_2 + \rho = 0. \tag{5}$$

Using the definition of breaks, the inequalities $\text{br}(y, E_1, f) \geq 2$ and $\text{br}(y, E_2, f) \geq 2$ imply

$$0 = (f \circ x)'(t_y) = \nabla f(y)^T x'(t_y),$$
$$0 = (f \circ \bar{x})'(\bar{t}_y) = \nabla f(y)^T \bar{x}'(\bar{t}_y),$$

where $x'(t_y)$ and $\bar{x}'(\bar{t}_y)$ are tangent vectors at y along $\mathrm{bd}(E_1)$ and $\mathrm{bd}(E_2)$. Since $y \in \mathrm{vert}(F)$, the simplifying assumption of section 1.1 implies that $x'(t_y)$ and $\bar{x}'(\bar{t}_y)$ are linearly independent. So $\nabla f(y) = 0$, i.e.,

$$y_1 R_{11} + y_2 R_{21} + r_1 = 0, \tag{6}$$

$$y_1 R_{21} + y_2 R_{22} + r_2 = 0. \tag{7}$$

Considering (R, r, ρ) to be unknown, (5)–(7) thus provide three homogeneous equations in (R, r, ρ). Moreover, these three equations can easily be seen to be linearly independent. Since R is symmetric, there are a total of six unknowns. So the space of solutions in (R, r, ρ) satisfying (5)–(7) has dimension three.

Table 2 provides three solutions (R, r, ρ) as suggested in the decomposition (4) of f. Specifically, each of the three component functions $T_{yE_1}^2$, $T_{yE_1}T_{yE_2}$, and L_{yz}^2 has a break at least 2 at y with respect to both $\mathrm{bd}(E_1)$ and $\mathrm{bd}(E_2)$. It remains to show that the three solutions are independent.

Suppose (4) satisfies $f = 0$, and define the affine function $M := \alpha_1 T_{yE_1} + \alpha_2 T_{yE_2}$. Then $f = T_{yE_1} M + \alpha_3 L_{yz}^2 = 0$. Since $T_{yE_1} \neq 0$, it is clear that $\alpha_3 = 0$ and then $M = 0$. Since T_{yE_1} and T_{yE_2} are clearly independent, $\alpha_1 = \alpha_2 = 0$ as well. ∎

PROPOSITION 3.7. *Let $f \neq 0$ and distinct y, z, $w \in N \cap \mathrm{bd}(F)$ be given. Suppose $y \in \mathrm{vert}(F)$. If $\mathrm{br}(y, E_1, f) \geq 2$ and $\mathrm{br}(y, E_2, f) \geq 2$, then $f = \bar{\alpha} L_{yz} L_{zw}$ for some $\bar{\alpha} \in \mathbb{R}$. As a consequence, f is not valid.*

Proof. Apply Lemma 3.6 to write f as (4). As $T_{yE_1}(z) \neq 0$ and

$$0 = 0 - 0 = f(z) - \alpha_3 L_{yz}^2(z) = T_{yE_1}(z) \left(\alpha_1 T_{yE_1}(z) + \alpha_2 T_{yE_2}(z) \right),$$

we have $(\alpha_1 T_{yE_1} + \alpha_2 T_{yE_2})(z) = 0$. Note that $\alpha_1 T_{yE_1} + \alpha_2 T_{yE_2}$ corresponds to a line passing through y and z. So there exists $\tilde{\alpha} \in \mathbb{R}$ such that $\alpha_1 T_{yE_1} + \alpha_2 T_{yE_2} = \tilde{\alpha} L_{yz}$. Now

$$f = \tilde{\alpha} T_{yE_1} L_{yz} + \alpha_3 L_{yz}^2 = L_{yz}(\tilde{\alpha} T_{yE_1} + \alpha_3 L_{yz}).$$

Using $L_{yz}(w) \neq 0$ and a similar argument as just applied, there exists $\bar{\alpha} \in \mathbb{R}$ such that $\tilde{\alpha} T_{yE_1} + \alpha_3 L_{yz} = \bar{\alpha} L_{yw}$. Then $f = \bar{\alpha} L_{yz} L_{yw}$, and $f \neq 0$ implies that $\bar{\alpha} \neq 0$. Since the lines L_{yz} and L_{yw} geometrically divide F into three parts and f cannot have the same sign on all three parts, f is not valid. ∎

PROPOSITION 3.8. *Let $f \neq 0$ and $y \in \mathrm{vert}(F)$ be given. If $br(y, E_1, f) \geq 3$ and $br(y, E_2, f) \geq 2$, then there exist $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $f = \alpha_1 T_{yE_1}^2 + \alpha_2 T_{yE_1} T_{yE_2}$. In addition, if there exists $z \in N \cap \mathrm{bd}(F)$ with $z \neq y$, then $f = \hat{\alpha} T_{yE_1} L_{yz}$ for some $\hat{\alpha} \in \mathbb{R}$, and as a consequence, f is not valid.*

Proof. Apply Lemma 3.6 to write f as (4). Note that $\mathrm{br}(y, E_1, T_{yE_1}^2) = 4$, $\mathrm{br}(y, E_1, T_{yE_1} T_{yE_2}) = 3$, and $\mathrm{br}(y, E_1, L_{yz}^2) = 2$ by Table 2. Proposition 3.3(i) thus implies $\alpha_3 = 0$. Rewriting f as $T_{yE_1}(\alpha_1 T_{yE_1} + \alpha_2 T_{yE_2})$, we can use the same technique as in the proof of Proposition 3.7 to prove the second statement of the proposition. ∎

Next, Lemma 3.9 and Proposition 3.10 allow us to characterize quadratics with specific breaks at zeros on both ellipsoids.

LEMMA 3.9. *Let f and feasible $y \in \mathrm{bd}(E_1) \cap N$ and feasible $z \in \mathrm{bd}(E_2) \cap N$ with $y \neq z$ be given. Then there exist $\beta_1, \beta_2, \beta_3, \beta_4$ such that*

$$f = \beta_1 T_{yE_1} T_{zE_2} + \beta_2 L_{yz}^2 + \beta_3 T_{zE_2} L_{yz} + \beta_4 T_{yE_1} L_{yz}. \tag{8}$$

Proof. We take the same approach as the proof of Lemma 3.6, but in this case, the equations $f(y) = f(z) = 0$ are just two independent equations that limit the dimension of solutions f to 4. The four functions $T_{yE_1} T_{zE_2}$, L_{yz}^2, $T_{zE_2} L_{yz}$, and $T_{yE_1} L_{yz}$ are clearly solutions, so (8) holds as long as the four are independent.

So suppose that f as presented in (8) satisfies $f = 0$, and define the affine function $M := \beta_2 L_{yz} + \beta_3 T_{zE_2} + \beta_4 T_{yE_1}$. Then $f = L_{yz} M + \beta_1 T_{yE_1} T_{zE_2}$. It is clear that $L_{yz} M$ and $\beta_1 T_{yE_1} T_{zE_2}$ are linearly dependent if and only if $M = 0$ and $\beta_1 = 0$. Thus, $0 = M(y) = \beta_3 T_{zE_2}(y)$, which implies $\beta_3 = 0$ since $T_{zE_2}(y) \neq 0$. Similarly $\beta_4 = 0$, and so finally $\beta_2 = 0$ as well. ∎

Proposition 3.10. *Let f and feasible $y \in \mathrm{bd}(E_1) \cap N$ and feasible $z \in \mathrm{bd}(E_2) \cap N$ with $y \neq z$ be given. If $\mathrm{br}(y, E_1, f) \geq 2$ and $\mathrm{br}(z, E_2, f) \geq 2$, then there exist β_1, β_2 such that $f = \beta_1 T_{yE_1} T_{zE_2} + \beta_2 L_{yz}^2$.*

Proof. Apply Lemma 3.9 to write f as (8). From Table 2, $\mathrm{br}(z, E_2, T_{yE_1} L_{yz}) = 1$, $\mathrm{br}(z, E_2, T_{yE_1} T_{zE_2}) = \mathrm{br}(z, E_2, L_{yz}^2) = 2$, and $\mathrm{br}(z, E_2, T_{zE_2} L_{yz}) = 3$. As $\mathrm{br}(z, E_2, f) \geq 2$, it holds that $\beta_4 = 0$ by Lemma 3.3(i). By symmetry, $\beta_3 = 0$ as desired. ∎

4. Global analysis of valid quadratics.

As discussed in section 2, Proposition 2.2 is the key result required for our choice of $\mathcal{G}$. In this section, we argue in Theorem 4.3 that Proposition 2.2 does indeed hold, i.e., we show that every quadratic function $f \in \mathcal{K}$ can be minorized by some $g \in \mathcal{G}$.

The following lemma gives conditions under which a valid $f \in \mathcal{K}$ can be perturbed to a valid $f + \lambda g$ for some $\lambda < 0$, where g is also valid. In other words, f can be minorized by $-\lambda g$. The key insight is to compare the zeros and breaks of f and g.

Lemma 4.1. *Let $f, g \in \mathcal{K}$, and suppose $\mathrm{Hess}(f) \notin \mathcal{S}_+^2$ and $N(f) \subseteq N(g)$ with $|N(f)|$ finite. In particular, $N(f) \subseteq \mathrm{bd}(F)$. Suppose that $\mathrm{br}(y, E_i, f) \leq \mathrm{br}(y, E_i, g)$ also holds for all $y \in \mathrm{bd}(E_i) \cap F \cap N(f)$, $i = 1, 2$. Then $f + \lambda g$ is valid for some $\lambda < 0$.*

Proof. Since the Hessian of f is not positive semidefinite, there exists a small $\lambda_1 < 0$ such that $\mathrm{Hess}(f + \lambda_1 g) \notin \mathcal{S}_+^2$. We will require $\lambda_1 \leq \lambda < 0$, in which case $f + \lambda g$ will attain its global minimum over F in the boundary $\mathrm{bd}(F)$.

Next let $y \in \mathrm{bd}(E_i) \cap F \cap N(f)$. Suppose $r = \mathrm{br}(y, E_i, f) \leq \mathrm{br}(y, E_i, g)$. In the intersection of F, $\mathrm{bd}(E_i)$, and a sufficiently small open neighborhood $O(y) \subseteq \mathbb{R}^2$ of y, we have the Taylor approximations

$$(f \circ x)(t) = \tfrac{1}{r!} \cdot (f \circ x)^{(r)}(t_y) \cdot (t - t_y)^r + \mathcal{O}((t - t_y)^{r+1}),$$
$$(g \circ x)(t) = \tfrac{1}{r!} \cdot (g \circ x)^{(r)}(t_y) \cdot (t - t_y)^r + \mathcal{O}((t - t_y)^{r+1}),$$

where $x(t)$ is any parameterization of $\mathrm{bd}(E_i)$, $y = x(t_y)$, and $\mathcal{O}((t - t_y)^{r+1})$ expresses terms of $t - t_y$ with degree at least $r + 1$. Since $(f \circ x)^{(r)}(t_y) \neq 0$, there exists a small $\lambda_y < 0$ such that $((f + \lambda_y g) \circ x)^{(r)}(t_y)$ is nonzero with the same sign as $(f \circ x)^{(r)}(t_y)$. Therefore, $f(z) + \lambda_y g(z) \geq 0$ for all $z \in F \cap \mathrm{bd}(E_i) \cap O(y)$, because $(f + \lambda_y g) \circ x$ and $f \circ x$ have the same local behavior around t_y. In words, $f + \lambda_y g$ is locally valid around y. We will also require $\lambda_y \leq \lambda < 0$.

Now consider $f + \lambda g$ over the complement $Q := \mathrm{bd}(F) \setminus \cup_{y \in N(f)} O(y)$. Because $\{O(y)\}$ is a finite collection of open sets containing the zeros of f, Q is compact and $\min_{x \in Q} f(x)$ is positive. Hence, there exists $\lambda_Q < 0$ such that $f + \lambda_Q g$ is valid over Q. We will also require $\lambda_Q \leq \lambda < 0$.

Based on the previous three paragraphs, we take λ to be the maximum of λ_1, λ_Q, and λ_y for all $y \in N(f)$. This proves the existence of $\lambda < 0$ such that $f + \lambda g$ is valid. ∎

For a valid $f \in \mathcal{K}$ with $\mathrm{Hess}(f) \notin \mathcal{S}^2_+$, nonvertex zeros, i.e., zeros in $\mathrm{bd}(F) \setminus \mathrm{vert}(F)$, have different break properties compared to vertex zeros. In particular, the following result shows that nonvertex zeros have even breaks.

LEMMA 4.2. *Let $f \in \mathcal{K}$ with $\mathrm{Hess}(f) \notin \mathcal{S}^2_+$, and suppose $y \in N(f)$ and $y \in \mathrm{bd}(E_1) \setminus \mathrm{vert}(F)$. Then $\mathrm{br}(y, E_1, f)$ is even, and in particular, $\mathrm{br}(y, E_1, f) \geq 2$.*

Proof. Let $x(t)$ be the parameterization of $\mathrm{bd}(E_1)$ such that $y = x(t_y)$. Since f is valid, $f(y) = 0$, and $y \notin \mathrm{vert}(F)$, the 1-dimensional function $(f \circ x)(t)$ has a local minimum at t_y in an open neighborhood containing t_y. Using standard calculus, this implies $\mathrm{br}(y, E_1, f)$ is even, and thus, no less than 2. ∎

We are finally ready to state our main theorem that Proposition 2.2 holds for our choice of $\mathcal{G}$.

THEOREM 4.3. *Every $f \in \mathcal{K}$ satisfies $f \succeq g$ for some $g \in \mathcal{G}$, where $\mathcal{G}$ is given by Table* 1.

Proof. If $\mathrm{Hess}(f) \in \mathcal{S}^2_+$, then the theorem holds true as $f \succeq f$ and $f \in \mathcal{G}$. So assume $\mathrm{Hess}(f) \notin \mathcal{S}^2_+$, in which case $N := N(f) \subseteq \mathrm{bd}(F)$. We define

$$\max \mathrm{br}(f) := \max\{\mathrm{br}(\hat{y}, E_i, f) : \hat{y} \in \mathrm{bd}(E_i) \cap F, i = 1, 2\},$$

possibly $-\infty$ (if $|N| = 0$) or ∞. That is, $\max \mathrm{br}(f)$ is the maximum break of f at its zeros measured along the corresponding ellipsoid boundaries. Choose any $y \in E_i$ such that $\max \mathrm{br}(f) = \mathrm{br}(y, E_i, f)$. Without loss of generality, we assume $i = 1$. Then define

$$\max 2\,\mathrm{br}(f) := \max\{\mathrm{br}(\hat{z}, E_2, f) : \hat{z} \in \mathrm{bd}(E_2) \cap F\},$$

possibly $-\infty$ (if E_2 has no zeros) or ∞. That is, $\max 2\,\mathrm{br}(f)$ is the maximum break of f measured with respect to the ellipsoid at which $\max \mathrm{br}(f)$ is not obtained. Also let $z \in E_2$ satisfy $\max 2\,\mathrm{br}(f) = \mathrm{br}(z, E_2, f)$. Note that $\max 2\,\mathrm{br}(f) \leq \max \mathrm{br}(f)$. The proof of the theorem considers cases based on $|N|$, $\max \mathrm{br}(f)$, and $\max 2\,\mathrm{br}(f)$.

First suppose $\max 2\,\mathrm{br}(f) \leq 1$. The contrapositive of Lemma 4.2 implies that $z \in \mathrm{vert}(F)$, which in turn implies $N \subseteq \mathrm{bd}(E_1)$. If $|N| = \infty$, then $f \in \mathcal{G}$ as a nonnegative multiple of $g_{E_1} \in \mathcal{G}$. If $|N| < \infty$ (including $N = \emptyset$), applying Lemma 4.1 with the breaks of g_{E_1} in Table 2, we see that f is minorized by $\lambda g_{E_1} \in \mathcal{G}$ for some $\lambda > 0$, as desired.

So we may assume $\max \mathrm{br}(f) \geq \max 2\,\mathrm{br}(f) \geq 2$. We consider two cases: (i) $z \neq y$; (ii) $z = y$. For (i), Proposition 3.10 implies $f = \beta_1 T_{yE_1} T_{zE_2} + \beta_2 L^2_{yz}$ for some $\beta_1, \beta_2 \in \mathbb{R}$. If $\beta_1 = 0$, then $\beta_2 \geq 0$ because $f \in \mathcal{K}$, but this contradicts our assumption that $\mathrm{Hess}(f) \notin \mathcal{S}^2_+$. So $\beta_1 \neq 0$, and in fact, $\beta_1 > 0$, otherwise, f would be invalid along the line L_{yz}. Hence, after scaling to $\beta_1 = 1$, f is minorized by a lifted-RLT member of $\mathcal{G}$.

For case (ii), $y = z \in \mathrm{vert}(F)$, and we consider two subcases: (a) $\max \mathrm{br}(f) \geq 3$, and (b) $\max \mathrm{br}(f) = 2$. For subcase (a), Proposition 3.8 implies that $f = \alpha_1 T^2_{yE_1} + \alpha_2 T_{yE_1} T_{yE_2}$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$. If either α_1 or α_2 is zero, then $f \in \mathcal{K}$ implies that the other parameter is nonnegative. Since we assume $\mathrm{Hess}(f) \notin \mathcal{S}^2_+$, the only possible case is that $\alpha_1 = 0 < \alpha_2$, so f is generated by a vertex RLT quadratic in $\mathcal{G}$. If both α_1 and α_2 are nonzero, then by the first part of Proposition 3.3 and the break information of $T^2_{yE_1}$ and $T_{yE_1} T_{yE_2}$ shown in Table 2, $\mathrm{br}(y, E_1, f) = 3$ and $\mathrm{br}(y, E_2, f) = 2$. Since the second part of Proposition 3.8 implies that f is not valid unless $|N| = 1$, f then can be minorized by a positive multiple of $T^2_{yE_1}$ by appealing to Lemma 4.1.

Finally, for subcase (b), the contrapositive of Proposition 3.7 implies that $|N| \le 2$. Then by Lemma 4.1, f can be minorized by a positive multiple of L_{yz}^2, which completes the proof. ∎

5. The separation problem and computational results. The goal of this section is to demonstrate how to use the lifted-RLT constraints of Table 1 in practice. Since there are an infinite number of such constraints, the separation problem is key. We first discuss separation of lifted-RLT constraints when $n = 2$ and then extend the technique to $n > 2$. We end the section with some computational tests.

Given a point $(\bar{x}, \bar{X})$, which satisfies some lifted-RLT constraints, we envision separation—finding a lifted-RLT constraint violating $(\bar{x}, \bar{X})$—as consisting of two steps. First choose $y \in \mathrm{bd}(E_1)$ and $z \in \mathrm{bd}(E_2)$. Then calculate the minimal $\lambda_{\min} < 0$ such that $T_{yE_1} T_{zE_2} + \lambda_{\min} L_{yz}^2$ is valid. We begin by discussing the calculation of $\lambda_{\min}$.

5.1. For $n = 2$, calculating $\lambda_{\min}$ given (y, z). For $n = 2$ and given $y \in \mathrm{bd}(E_1)$ and $z \in \mathrm{bd}(E_2)$, we argue that $\lambda_{\min}$ can be calculated with high precision.

Recall the definitions of T_{yE_1}, T_{zE_2}, and L_{yz} given at the beginning of section 2.3, and for any $\lambda < 0$, define

$$q_\lambda := T_{yE_1} T_{zE_2} + \lambda L_{yz}^2.$$

Note that $\mathrm{Hess}(q_\lambda) \notin \mathcal{S}_+^2$, so that the global minimizers of q_λ over F are contained in $\mathrm{bd}(F)$. Since $\mathrm{bd}(F) \subseteq \mathrm{bd}(E_1) \cup \mathrm{bd}(E_2)$, it follows that q_λ is valid over F if and only if it is simultaneously valid over both $\mathrm{bd}(E_1) \cap F$ and $\mathrm{bd}(E_2) \cap F$. We next will argue that the validity of $\mathrm{bd}(E_i) \cap F$ for each $i = 1, 2$ can be determined easily, so that checking validity of q_λ over F is easy. As a consequence, a simple bisection procedure over λ can be used to calculate $\lambda_{\min}$.

We discuss only validity over $\mathrm{bd}(E_1) \cap F$ since the second case is similar. Also, for simplification but without loss of generality, we assume E_1 equals the unit ball $\mathcal{B} := \{x \in \mathbb{R}^2 : x^T x \le 1\}$. The following lemma provides the key insight for our approach. Note that, when applying the lemma below, w will play a role different than z, although w could be equal to z.

Lemma 5.1. *Suppose $y, w \in \mathrm{bd}(\mathcal{B})$ with $y \ne w$. Then $T_{y\mathcal{B}}(x) T_{w\mathcal{B}}(x) = L_{yw}^2(x)$ for all $x \in \mathrm{bd}(\mathcal{B})$ when $n = 2$.*

Proof. In this case, $T_{y\mathcal{B}}(x) = 1 - y^T x$ and $T_{w\mathcal{B}}(x) = 1 - w^T x$. Also $L_{yw}^2(x) = (u^T(x - y))^2$, where u is a unit vector that is perpendicular to $w - y$. We take $u = (y + w)/\|y + w\|$, and by an orthogonal rotation, we assume without loss of generality that $y = (1, 0)^T$. Assuming $x^T x = 1$ and using $y^T y = w^T w = 1$, we have

$$\begin{aligned} L_{yw}^2(x) = (u^T(x - y))^2 = \frac{((y + w)^T(x - y))^2}{\|y + w\|^2} &= \frac{((1 + w_1)(x_1 - 1) + w_2 x_2)^2}{2(1 + w_1)} \\ &= \frac{1}{2}(1 + w_1)(x_1 - 1)^2 + w_2(x_1 - 1)x_2 + \frac{(1 - w_1^2)(1 - x_1^2)}{2(1 + w_1)} \\ &= (1 - x_1)\left(\frac{1}{2}(1 + w_1)(1 - x_1) - w_2 x_2 + \frac{1}{2}(1 - w_1)(1 + x_1)\right) \\ &= (1 - x_1)(1 - w_1 x_1 - w_2 x_2) = (1 - y^T x)(1 - w^T x) \\ &= T_{y\mathcal{B}}(x) T_{w\mathcal{B}}(x). \end{aligned}$$ ∎

By construction, the line L_{yz} passes through $y \in \mathrm{bd}(\mathcal{B})$ and $z \in \mathrm{bd}(E_2)$. Geometrically, L_{yz} must also intersect $\mathrm{bd}(\mathcal{B})$ in a second point, say $w \in \mathrm{bd}(\mathcal{B})$ (w may equal z when $z \in \text{vert}(F)$). In addition, $L_{yz} = L_{yw}$. Then Lemma 5.1 shows that

$$x \in \text{bd}(\mathcal{B}) \quad \Longrightarrow \quad \begin{aligned} q_\lambda(x) &= T_{y\mathcal{B}}(x)T_{zE_2}(x) + \lambda L_{yz}^2(x) \\ &= T_{y\mathcal{B}}(x)T_{zE_2}(x) + \lambda L_{yw}^2(x) \\ &= T_{y\mathcal{B}}(x)T_{zE_2}(x) + \lambda T_{y\mathcal{B}}(x)T_{w\mathcal{B}}(x) \\ &= T_{y\mathcal{B}}(x) \cdot \big(T_{zE_2}(x) + \lambda T_{w\mathcal{B}}(x)\big). \end{aligned}$$

In words, q_λ restricted to $\text{bd}(\mathcal{B})$ can be expressed as the product of two linear functions, $T_{y\mathcal{B}}$ and $l_\lambda := T_{zE_2} + \lambda T_{w\mathcal{B}}$. Since $T_{y\mathcal{B}}$ is valid over $\text{bd}(\mathcal{B}) \cap F$ and zero only at a single point, q_λ is valid over $\text{bd}(\mathcal{B}) \cap F$ if and only if l_λ is valid over $\text{bd}(\mathcal{B}) \cap F$, that is, if and only if

$$(S_\lambda^1) \qquad v(S_\lambda^1) \;:=\; \begin{array}{ll} \min & l_\lambda(x) \\ \text{s.t.} & x^T x = 1,\, x \in E_2 \end{array}$$

is nonnegative. So we have reduced the validity of q_λ to the validity of l_λ over $\text{bd}(\mathcal{B}) \cap F$.

We claim that, in turn, the validity of l_λ holds if and only if the optimal value of

$$(S_\lambda^2) \qquad v(S_\lambda^2) \;:=\; \begin{array}{ll} \min & 1 - x^T x \\ \text{s.t.} & l_\lambda(x) \le 0,\, x \in E_2 \end{array}$$

is nonnegative.

PROPOSITION 5.2. *For all $\lambda \le 0$, $v(S_\lambda^1) \ge 0$ if and only if $v(S_\lambda^2) \ge 0$.*

Proof. ($\Leftarrow$ contrapositive) If $v(S_\lambda^1) < 0$, then there exists $x \in \text{bd}(\mathcal{B}) \cap E_2$ such that $l_\lambda(x) < 0$. We consider two cases: $x \in \text{int}(E_2)$, and $x \in \text{bd}(E_2)$. In the first case, we can perturb x to $\hat{x}$ such that $\hat{x}^T\hat{x} > 1$, $\hat{x} \in \text{int}(E_2)$, and $l_\lambda(\hat{x}) < 0$. This implies $v(S2) < 0$. In the second case, $x \in \text{vert}(F)$. We can then perturb x to $\hat{x}$ such that $\hat{x} \in \text{bd}(\mathcal{B})$, $\hat{x} \in \text{int}(E_2)$, and $l_\lambda(\hat{x}) < 0$. Then the first case applies to $\hat{x}$.

($\Rightarrow$) Define the convex feasible set of (S_λ^2) to be $R_\lambda^2 := \{x : l_\lambda(x) \le 0\} \cap E_2$. If $R_\lambda^2 \subseteq \mathcal{B}$, then $v(S_\lambda^2) \ge 0$. So suppose $R_\lambda^2 \nsubseteq \mathcal{B}$ and consider two subcases: (i) R_λ^2 crosses $\text{bd}(\mathcal{B})$; (ii) R_λ^2 is completely outside of $\text{int}(\mathcal{B})$. For subcase (i), R_λ^2 must be full dimensional. So we clearly have points satisfying $x \in \text{bd}(\mathcal{B})$, $l_\lambda(x) < 0$, and $x \in E_2$. However, this is inconsistent with the assumption $v(S_\lambda^1) \ge 0$.

For subcase (ii), we consider three mutually exclusive and collectively exhaustive alternatives: (a) $\lambda < 0$ and $T_{w\mathcal{B}}(z) = 0$; (b) $\lambda < 0$ and $T_{w\mathcal{B}}(z) > 0$; and (c) $\lambda = 0$. If (a), then $z \in \text{vert}(F)$ and $w = z$. Note $\{x : l_\lambda(x) = 0\} \subseteq \{x : T_{zE_2}(x)T_{w\mathcal{B}}(x) \ge 0\} = TC_z(F) \cup -TC_z(F)$, where $TC_z(F)$ is the tangent cone of F at z. Then $\lambda < 0$ implies that l_λ intersects $\text{int}(F)$, a contradiction. If (b), then l_λ evaluated at z equals $T_{zE_2}(z) + \lambda T_{w\mathcal{B}}(z) = \lambda T_{w\mathcal{B}}(z) < 0$. Then we can perturb z to $\hat{z}$ such that $\hat{z} \in \text{int}(F)$ and $l_\lambda(z) < 0$, again a contradiction. So in fact (c) is the only true alternative, in which case $l_\lambda = T_{zE_2}$, $R_\lambda^2 = \{z\}$, and $v(S_\lambda^2) \ge 0$. ∎

Note that calculating $v(S_\lambda^2)$ is a TRS with one linear constraint, which has been proved tractable in [22].

To illustrate Proposition 5.2 and the calculation of $\lambda_{\min}$, we consider a geometric example in Figure 2. Let R_λ^1 and R_λ^2 be the feasible regions of (S_λ^1) and (S_λ^2), which are bold and shaded, respectively. In the leftmost picture, $\lambda = -0.3$, and $v(S_\lambda^1) > 0$ because R_λ^1 lies entirely on the nonnegative side of l_λ, while $v(S_\lambda^2) > 0$ because

$R_\lambda^2 \subseteq \operatorname{int}(\mathcal{B})$. As λ decreases, l_λ rotates clockwise around point p, and first intersects R_λ^1 at point q, as shown in the middle picture (in which $\lambda = -0.37$). For this λ, (S_λ^1) and (S_λ^2) have the same minimizer q and minimum value 0, and moreover $\lambda = \lambda_{\min}$. If λ continues to decrease past $\lambda_{\min}$, then q lies on the negative side of l_λ, and thus $v(S_\lambda^1) < 0$ as shown in the rightmost picture with $\lambda = -0.46$. Also, since $R_\lambda^2 \setminus \mathcal{B} \neq \emptyset$, it holds that $v(S_\lambda^2) < 0$.

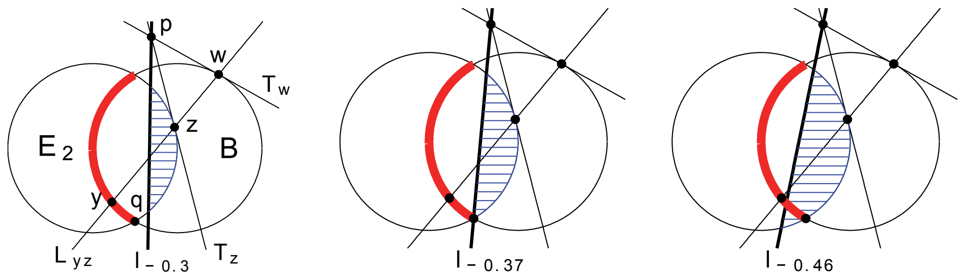

FIG. 2. *An example to illustrate Proposition* 5.2.

5.2. For any n, choosing (y, z). Now we propose a heuristic way to choose $y \in \operatorname{bd}(E_1)$ and $z \in \operatorname{bd}(E_2)$ upon which to base the lifted-RLT constraint of the previous subsection. Our idea is based on looking for a violated SOCRLT constraint as described in [8]. Note that, by the discussion at the end of section 2.1, if there exists a violated SOCRLT, then there exists a violated RLT constraint, in which case there also exists a violated lifted-RLT constraint. On the other hand, the converse does not hold, and accordingly we only propose this procedure when a violated SOCRLT is found.

Suppose that $(\bar{x}, \bar{X})$ is our current solution. Based on $(\bar{x}, \bar{X})$ we solve the SOCRLT separation problem as discussed in section 5 of [8]. If a violated SOCRLT is found, we use the solution of the separation problem to choose y. In particular, the separation problem always yields a distinguished $y \in \operatorname{bd}(E_1)$, which is the "support point" of the SOCRLT constraint. Moreover, adding the violated SOCRLT to the current relaxation and resolving guarantees that the new SOCRLT subsequently becomes active, which in turn yields a $z \in \operatorname{bd}(E_2)$. In terms of the discussion in section 2 and the SOCRLT constraint (2), the formula for z is

$$z := \frac{\beta\,\hat{x} - \hat{X}\alpha}{\beta - \alpha^T\hat{x}},$$

where $(\hat{x}, \hat{X})$ is optimal after the new SOCRLT constraint has been added. Please note that the SOCRLT constraints are used only for generating (y, z) and are never added to the current relaxation.

5.3. When $n > 2$. We next discuss a generalization of the lifted-RLT constraints for general n. Let $y \in \operatorname{bd}(E_1) \cap F$ and $z \in \operatorname{bd}(E_2) \cap F$ with $y \neq z$ be given. To generalize the function $L_{yz}^2(x)$ in dimension 2, our idea is to consider functions of the type $M_{yz}(x) := (x - z)^T H(x - z)$, where H satisfies $H \succeq 0$ and $(y - z)^T H(y - z) = 0$. This ensures that $M_{yz}(x) \geq 0$ and $M_{yz}(y) = M_{yz}(z) = 0$ in analogy with $L_{yz}^2(x)$. Then with y, z, and M chosen, we search for the most negative $\lambda_{\min} < 0$ such that

$$T_{yE_1} T_{zE_2} + \lambda_{\min} M_{yz} \geq 0 \tag{9}$$

is valid for F. We are unsure whether this class of lifted-RLT constraints closes the relaxation gap for general n, but we propose the following heuristic to generate such cuts in practice.

For a given current solution $(\bar{x}, \bar{X})$, we choose (y, z) exactly as described in section 5.2, which works for general n. Then we search for a matrix H such that $2H$ will serve as the Hessian of M_{yz}. To guide our choice of H, we first examine the linearized form of (9):

$$\left[\beta\delta - \beta\gamma^T x - \delta\alpha^T x + \alpha^T X \gamma\right] + \lambda_{\min}\left[H \bullet X - 2y^T H x + y^T H y\right] \geq 0,$$

where $T_{yE_1}(x) := \beta - \alpha^T x$ and $T_{zE_2}(x) := \delta - \gamma^T x$. Given this form and keeping in mind that $\lambda_{\min} < 0$ is yet to be determined—it will depend on H—a reasonable choice for H is one that maximizes $H \bullet \bar{X} - 2y^T H \bar{x} + y^T H y$. This will increase the chance that the linearized form is ultimately violated when plugging in $(\bar{x}, \bar{X})$ and $\lambda_{\min}$, i.e., that we will be able to find a good cut. So we solve

$$\begin{aligned}
\max_{H} \quad & H \bullet \bar{X} - 2y^T H \bar{x} + y^T H y \\
\text{s.t.} \quad & H(y - z) = 0, \\
& \operatorname{trace}(H) = 1, \\
& H \succeq 0.
\end{aligned}$$

Because $H \succeq 0$, the constraint $H(y - z) = 0$ is equivalent to $(y - z)^T H (y - z) = 0$. Also, the normalization constraint $\operatorname{trace}(H) = 1$ simply bounds the feasible region.

Given y, z, and H, it remains to calculate $\lambda_{\min}$. Unfortunately, for general n, we do not know how to calculate $\lambda_{\min}$ exactly, but we can calculate an upper bound $\lambda_{\min} \leq \lambda_{\text{upper}} < 0$ as follows. Without loss of generality, we assume that E_1 equals the unit ball $\mathcal{B}$, and recall that when $n = 2$, Lemma 5.1 allows us to rewrite $L_{yz}^2 = T_{y\mathcal{B}} T_{w\mathcal{B}}$ in the restricted domain $\operatorname{bd}(\mathcal{B})$. When $n > 2$, simple examples show that the analog of Lemma 5.1 does not hold. However, we try a similar idea by looking for $\alpha \geq 1$ such that $M_{yz}(x) \leq \alpha T_{y\mathcal{B}}(x) T_{w\mathcal{B}}(x)$ for all $x \in \operatorname{bd}(\mathcal{B})$, where $w \in \operatorname{bd}(\mathcal{B})$ lies on the line connecting y and z (just as in section 5.1). Note that $M_{yz}(w) = 0$. In fact, the smallest such $\alpha_{\min}$ can be calculated by bisection on α using the solution of the following (equality constrained) TRS problem:

$$\begin{aligned}
\min \quad & \alpha T_{y\mathcal{B}}(x) T_{w\mathcal{B}}(x) - M_{yz}(x) \\
\text{s.t.} \quad & x \in \operatorname{bd}(\mathcal{B}).
\end{aligned}$$

The basic decision in the bisection routine is as follows: if the optimal value is negative, then we decrease α; otherwise, we increase α. After $\alpha_{\min}$ is determined, we then follow the ideas given in section 5.1 to calculate a minimum $\lambda < 0$ which guarantees that $T_{y\mathcal{B}}(T_{zE_2} + \lambda T_{w\mathcal{B}}) \geq 0$ is valid on F. Finally, we define $\lambda_{\text{upper}} := \lambda / \alpha_{\min}$ so that

$$\begin{aligned}
0 &\leq T_{y\mathcal{B}}(T_{zE_2} + \lambda T_{w\mathcal{B}}) \\
&= T_{y\mathcal{B}} T_{zE_2} + \lambda T_{y\mathcal{B}} T_{w\mathcal{B}} \\
&= T_{y\mathcal{B}} T_{zE_2} + \lambda_{\text{upper}} \alpha_{\min} T_{y\mathcal{B}} T_{w\mathcal{B}} \\
&\leq T_{y\mathcal{B}} T_{zE_2} + \lambda_{\text{upper}} M_{yz},
\end{aligned}$$

showing that $T_{y\mathcal{B}} T_{zE_2} + \lambda_{\text{upper}} M_{yz}$ is valid on F.

TABLE 3
Numerical results on TTRS instances from [8].

n	% solved by basic SDP	% solved by adding SOC-RLT cuts to basic SDP[1]	additional % solved by adding heuristic lifted-RLT cuts to basic SDP	% still unsolved
5	92.2	4.0	2.0	1.8
10	24.6	68.3	2.2	4.9
20	4.1	85.3	4.0	6.6

5.4. Computational tests. For $n = 2$, consider the following instance of TTRS with concentric ellipsoids centered at 0:

$$\begin{aligned} v^* \ := \ \min_{x\in\mathbb{R}^n} \ & x^T C x + 2\, c^T x \\ \text{s.t.} \ & x^T x \le 1, \\ & x^T A_2\, x \le 1, \end{aligned}$$

where

$$C = \begin{pmatrix} -3/5 & \sqrt{6}/4 \\ \sqrt{6}/4 & -2/5 \end{pmatrix}, \quad c = -\frac{1}{2}\begin{pmatrix} \sqrt{6}/2 \\ 1 \end{pmatrix}, \quad A_2 = \frac{1}{2}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since C is not positive semidefinite, we know that an optimal solution must occur on the boundary. By carefully tracing the boundary of the feasible region, we can verify that $x^* = (1,1)^T/\sqrt{2}$ is a global optimal solution with optimal value $v^* \approx -1.4608$. Numerical results show that the optimal value of the relaxation with only SOCRLT constraints is -1.5, a gap of 2.74%. However, if we construct a lifted-RLT constraint based on $y = (0,1)^T$ and $z = (\sqrt{6}/2, 0)^T$, we can calculate $\lambda_{\min} \approx 0.2986$. In fact, with some care, one can see that $\lambda_{\min}$ is determined by x^* being the third zero of the lifted-RLT quadratic along with y and z. In this case, we have the following precise formula for $\lambda_{\min}$:

$$\lambda_{\min} = \frac{20 + 5(\sqrt{6} - 2\sqrt{3} - 2\sqrt{2})}{18 + 4(\sqrt{6} - 2\sqrt{3} - 2\sqrt{2})} \approx 0.2986.$$

After adding the lifted-RLT constraint, we get the exact optimal value v^* with $X^* \approx x^*(x^*)^T$.

For $n = 2$, we also tested our approach on the example in section 5.2 of [8]. The relaxation gap is closed after six lifted-RLT cuts in the sense that the optimal solution $(1, x^T; x, X)$ has numerical rank 1.

For $n > 2$, we also solve the instances generated in [8], where 1,000 instances of (TTRS) were generated for each of $n = 5$, 10, 20. All tests were peformed on a Macintosh OS X desktop with 3.2 GHz Quad-core Intel Core i5 processor and 8 GB 1600 MHz DDR3 SDRAM using YALMIP [16], Matlab, and Mosek [2]. For the three different values of n, [8] found that 41, 70, and 104 instances could not be solved by adding SOCRLT constraints to the basic SDP relaxation. (In [8], an instance was regarded as solved if the relative gap between the relaxation value and the feasible value gotten by extracting x is less than 10^{-4}.) Applying our heuristic lifted-RLT constraints to the 215 previously unsolved instances, we can solve 82—about 38%—of them; see Table 3. However, the CPU time increases dramatically when separating the lifted-RLT cuts. For example, on a particular $n = 20$ instance, it takes 4.2 seconds

[1] This column is slightly different from [8] because we use a different SDP solver in the experiment.

to solve with only the SOCRLT constraints, while the same instance with lifted-RLT cuts requires 91.9 seconds. In general, our lifted-RLT process takes more than ten times longer on average than solving with only SOCRLT constraints.

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