Quadratic Programs with Hollows

Boshi Yang[*] Kurt Anstreicher[†] Samuel Burer[‡]

January 6, 2016

Abstract

Let $\mathcal{F}$ be a quadratically constrained, possibly nonconvex, bounded set, and let $\mathcal{E}_1, \ldots, \mathcal{E}_l$ denote ellipsoids contained in $\mathcal{F}$ with non-intersecting interiors. We prove that minimizing an arbitrary quadratic $q(\cdot)$ over $\mathcal{G} := \mathcal{F} \setminus \cup_{k=1}^{\ell} \operatorname{int}(\mathcal{E}_k)$ is no more difficult than minimizing $q(\cdot)$ over $\mathcal{F}$ in the following sense: if a given semidefinite-programming (SDP) relaxation for $\min\{q(x) : x \in \mathcal{F}\}$ is tight, then the addition of l linear constraints derived from $\mathcal{E}_1, \ldots, \mathcal{E}_l$ yields a tight SDP relaxation for $\min\{q(x) : x \in \mathcal{G}\}$. We also prove that the convex hull of $\{(x, xx^T) : x \in \mathcal{G}\}$ equals the intersection of the convex hull of $\{(x, xx^T) : x \in \mathcal{F}\}$ with the same l linear constraints.

Keywords: nonconvex quadratic programming, semidefinite programming, convex hull.

Mathematics Subject Classification: 90C20, 90C22, 90C25, 90C26, 90C30.

1 Introduction

Let

$$\mathcal{F} := \left\{ x \in \mathbb{R}^n : x^T A_i x + 2a_i^T x + \alpha_i \le 0 \ (i = 1, \ldots, m) \right\}$$

denote a bounded, full-dimensional, quadratically constrained set in $\mathbb{R}^n$, which may in general be nonconvex. Also, let $\mathcal{E}_k := \{x \in \mathbb{R}^n : x^T W_k x + 2w_k^T x + \omega_k \le 0\}$, for $k = 1, \ldots, l$, denote full-dimensional ellipsoids, each specified by a positive definite symmetric matrix $W_k \in \Re^{n \times n}$, vector $w_k \in \Re^n$ and scalar $\omega_k \in \mathbb{R}$. If each $\mathcal{E}_k \subseteq \mathcal{F}$ and the interiors of no two ellipsoids intersect, we say that the set

$$\mathcal{H} := \left\{ x \in \mathbb{R}^n : x^T W_k x + 2w_k^T x + \omega_k \ge 0 \ (k = 1, \ldots, l) \right\}$$

[*]Tepper School of Business, Carnegie Mellon University, Pittsburgh, PA, 15213, USA. Email: boshiy@andrew.cmu.edu.

[†]Department of Management Sciences, University of Iowa, Iowa City, IA, 52242-1994, USA. Email: kurt-anstreicher@uiowa.edu.

[‡]Department of Management Sciences, University of Iowa, Iowa City, IA, 52242-1994, USA. Email: samuel-burer@uiowa.edu.

induces non-intersecting hollows in $\mathcal{F}$. Geometrically, the set $\mathcal{G} := \mathcal{F} \cap \mathcal{H}$ results by deleting l disjoint, open ellipsoids from $\mathcal{F}$. In this note, we study the relationship between the two optimization problems

$$v(q, \mathcal{F}) := \min\{q(x) : x \in \mathcal{F}\}$$
$$v(q, \mathcal{G}) := \min\{q(x) : x \in \mathcal{G}\}$$

where $q(x) := x^T Q x + 2c^T x$ is a general, possibly nonconvex quadratic.

Optimizing $q(\cdot)$ over $\mathcal{G}$ certainly cannot be easier than optimizing $q(\cdot)$ over $\mathcal{F}$, and at least in some cases appears to be more difficult; for example, $\mathcal{G}$ is nonconvex even when $\mathcal{F}$ is convex. On the other hand, there are reasons to suspect that the complexity of optimizing $q(\cdot)$ over $\mathcal{G}$ should be closely related to that of optimizing $q(\cdot)$ over $\mathcal{F}$. To optimize over $\mathcal{G}$, one can first optimize over $\mathcal{F}$. If the resulting optimal x^* is in $\mathcal{G}$, then clearly x^* is optimal over $\mathcal{G}$. On the other hand, if $x^* \notin \mathcal{G}$, then because $\mathcal{H}$ induces non-intersecting hollows in $\mathcal{F}$, x^* must lie in the interior of $\mathcal{F}$ within exactly one of the deleted ellipsoids. It then follows that $q(\cdot)$ must be convex and that the global minimum over G is found on the boundary of that deleted ellipsoid, in which case the global minimum can be found by solving an instance of the equality-constrained *trust-region subproblem* [7]. Our note formalizes this intuition by studying semidefinite relaxations and reformulations of $v(q, \mathcal{F})$ and $v(q, \mathcal{G})$.

The most basic semidefinite-programming (SDP) relaxation of the set $\mathcal{F}$ is the *Shor relaxation*:

$$\mathcal{S}(\mathcal{F}) := \left\{ (x, X) : \begin{array}{c} A_i \bullet X + 2a_i^T x + \alpha_i \le 0 \ (i = 1, \ldots, m) \\ Y(x, X) \succeq 0 \end{array} \right\}$$

where

$$Y(x, X) := \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}$$

is an $(n+1) \times (n+1)$ symmetric matrix[1]. Note that $\mathcal{S}(\mathcal{F})$ may be an unbounded set even when $\mathcal{F}$ is bounded. On the other hand, the tightest convex relaxation of $\mathcal{F}$ in the space of variables (x, X) is the convex hull

$$\mathcal{C}(\mathcal{F}) := \operatorname{conv}\left\{(x, xx^T) : x \in \mathcal{F}\right\}$$

which is compact because $\mathcal{F}$ is. Clearly $\mathcal{C}(\mathcal{F}) \subseteq \mathcal{S}(\mathcal{F})$, and we call any closed, convex set $\mathcal{R}(\mathcal{F})$ a *valid SDP relaxation of* $\mathcal{F}$ if $\mathcal{C}(\mathcal{F}) \subseteq \mathcal{R}(\mathcal{F}) \subseteq \mathcal{S}(\mathcal{F})$.[2] In particular, both $\mathcal{C}(\mathcal{F})$ and

[1] More formally, $\text{proj}_x(\mathcal{S}(\mathcal{F}))$ is a relaxation of $\mathcal{F}$, where $\text{proj}_x(\cdot)$ denotes projection onto the x coordinates. We ignore this distinction between $\mathcal{S}(\mathcal{F})$ and $\text{proj}_x(\mathcal{S}(\mathcal{F}))$ to reduce notation.

[2] To be usable in practice, a valid SDP relaxation $\mathcal{R}(\mathcal{F})$ should have a known positive semidefinite (PSD)

$\mathcal{S}(\mathcal{F})$ are valid SDP relaxations of $\mathcal{F}$. Furthermore, any valid SDP relaxation $\mathcal{R}(\mathcal{F})$ of $\mathcal{F}$ gives rise to a relaxation of $v(q, \mathcal{F})$,

$$v(q, \mathcal{R}(\mathcal{F})) := \min\{Q \bullet X + 2c^T x : (x, X) \in \mathcal{R}(\mathcal{F})\}$$

such that $v(q, \mathcal{F}) \geq v(q, \mathcal{C}(\mathcal{F})) \geq v(q, \mathcal{R}(\mathcal{F})) \geq v(q, \mathcal{S}(\mathcal{F}))$. In fact, the first inequality is tight.

Proposition 1. *The equality $v(q, \mathcal{F}) = v(q, \mathcal{C}(\mathcal{F}))$ holds for all quadratic functions $q(\cdot)$.*

Proof. We have $v(q, \mathcal{C}(\mathcal{F})) \leq v(q, \mathcal{F})$ by construction. To show the reverse inequality, note that because $\mathcal{C}(\mathcal{F})$ is convex and the objective $Q \bullet X + 2c^T x$ is linear, a solution of the problem defining $v(q, \mathcal{C}(\mathcal{F}))$ must occur at an extreme point of $\mathcal{C}(\mathcal{F})$. However all extreme points of $\mathcal{C}(\mathcal{F})$ are of the form $(x, xx^T), x \in \mathcal{F}$. It follows that $v(q, \mathcal{C}(\mathcal{F})) = Q \bullet xx^T + 2c^T x = q(x)$ for some $x \in \mathcal{F}$, and therefore $v(q, \mathcal{F}) \leq v(q, \mathcal{C}(\mathcal{F}))$. $\square$

With respect to $\mathcal{G}$, we also define $\mathcal{S}(\mathcal{G})$, $\mathcal{C}(\mathcal{G})$, and $\mathcal{R}(\mathcal{G})$ similarly. Specifically, the Shor relaxation is

$$\mathcal{S}(\mathcal{G}) := \left\{ (x, X) : \begin{array}{c} A_i \bullet X + 2a_i^T x + \alpha_i \leq 0 \ (i = 1, \ldots, m) \\ W_k \bullet X + 2w_k^T x + \omega_k \geq 0 \ (k = 1, \ldots, l) \\ Y(x, X) \succeq 0 \end{array} \right\}$$

and we also write $\mathcal{S}(\mathcal{G}) = \mathcal{S}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$, where

$$\mathcal{L}(\mathcal{H}) := \left\{ (x, X) : W_k \bullet X + 2w_k^T x + \omega_k \geq 0 \ (k = 1, \ldots, l) \right\}.$$

We prove two main results. First, we show that for a valid SDP relaxation $\mathcal{R}(\mathcal{F})$, if the SDP optimal value $v(q, \mathcal{R}(\mathcal{F}))$ equals the original optimal value $v(q, \mathcal{F})$, then defining $\mathcal{R}(\mathcal{G}) := \mathcal{R}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$, the relaxed value $v(q, \mathcal{R}(\mathcal{G}))$ equals $v(q, \mathcal{G})$; see Theorem 1. In words, if an SDP relaxation has no gap over $\mathcal{F}$, then the SDP relaxation obtained by simply adding the l linear constraints $W_k \bullet X + 2w_k^T x + \omega_k \geq 0$ also has no gap over $\mathcal{G}$. Second, we establish that the convex hulls $\mathcal{C}(\mathcal{F})$ and $\mathcal{C}(\mathcal{G})$ are related according to the equation $\mathcal{C}(\mathcal{G}) = \mathcal{C}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$; see Corollary 1. That is, the same linear constraints $W_k \bullet X + 2w_k^T x + \omega_k \geq 0$ are precisely what is required to capture $\mathcal{C}(\mathcal{G})$ from $\mathcal{C}(\mathcal{F})$.

representation [11, Section 6.4]. However, it is convenient in this note to consider $\mathcal{R}(\mathcal{F})$ to be a valid SDP relaxation regardless of whether or not an explicit PSD representation for $\mathcal{R}(\mathcal{F})$ is known. We also apply this terminology to $\mathcal{C}(\mathcal{F})$, which in fact may not have an explicit PSD representaton—although the PSD constraint is always valid for $\mathcal{C}(\mathcal{F})$.

We provide two proofs of Corollary 1. The first proof depends on a third result proved in this note, which provides an alternative characterization of $\mathcal{C}(\mathcal{F})$ and has, to our knowledge, not appeared in the literature. The second proof, in contrast, connects better with existing proof techniques for studying convex hulls such as $\mathcal{C}(\mathcal{F})$. Finally, Section 3 provides counterexamples showing the necessity of the assumptions that each $\mathcal{E}_k \subseteq \mathcal{F}$ and that the interiors of $\{\mathcal{E}_k\}$ are non-intersecting.

Our results are related to a number of prior works concerning the tightness of SDP relaxations. It is well known that the Shor relaxation is tight in the convex programming case, corresponding to $A_i \succeq 0$, $i = 1, \ldots m$ and $Q \succeq 0$. A classical nonconvex problem with a tight Shor relaxation is the trust-region subproblem (TRS) [7, 14], whose feasible set is the unit ball with arbitrary $q(\cdot)$. The *generalized trust-region subproblem* [15] removes a concentric ball from the feasible set of TRS and yet still has a tight Shor relaxation obtained by adding a single linear constraint to the SDP relaxation of TRS [13, 17]. Other extensions to TRS are also known to have tight SDP relaxations, sometimes with additional valid inequalities added to the Shor relaxation: TRS with a single linear cut [5, 16]; TRS with multiple, non-intersecting linear cuts [6]; TRS with a homogeneous quadratic objective and an additional concentric, ellipsoidal constraint [17]; and TRS with an additional ellipsoidal constraint and satisfying various conditions on the quadratic function and/or at local minimizers [2, 10]. Many, but not all, of these results are based on characterizing the convex hull $\mathcal{C}(\mathcal{F})$ for the various feasible sets $\mathcal{F}$ under consideration. Characterizing $\mathcal{C}(\mathcal{F})$ has also been studied for some low-dimensional polyhedral $\mathcal{F}$, e.g., triangles and convex quadrilaterals in $\mathbb{R}^2$ and tetrahedra in $\mathbb{R}^3$ [1, 4]. Other authors have considered valid cuts of the form $\|x - c\|_2 \geq r$ for mixed-integer nonlinear programs [8] and valid linear cuts for the optimization of a convex quadratic over the deletion of an ellipsoid [3].

2 Exact Representations with Hollows

In this section, we present the main results of the note. The first theorem proves that a tight SDP relaxation of $v(q, \mathcal{F})$ gives rise to a tight relaxation of $v(q, \mathcal{G})$.

Theorem 1. *Let $\mathcal{R}(\mathcal{F})$ be a valid SDP relaxation of $\mathcal{F}$, and let $q(\cdot)$ be given. If $v(q, \mathcal{R}(\mathcal{F})) = v(q, \mathcal{F})$ and $\mathcal{H}$ induces non-intersecting hollows in $\mathcal{F}$, then $\mathcal{R}(\mathcal{G}) := \mathcal{R}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$ is a valid SDP relaxation of $\mathcal{G} := \mathcal{F} \cap \mathcal{H}$ and $v(q, \mathcal{R}(\mathcal{G})) = v(q, \mathcal{G})$.*

Proof. By construction, it is clear that $\mathcal{R}(\mathcal{G})$ is a valid SDP relaxation of $\mathcal{G}$. Since $q(\cdot)$ is fixed in this proof, we write $v(\cdot) := v(q, \cdot)$ for simplicity. Clearly $v(\mathcal{G}) \geq v(\mathcal{R}(\mathcal{G}))$ since $\mathcal{R}(\mathcal{G})$ is a valid relaxation of $\mathcal{G}$. So it remains to prove the reverse inequality.

If $v(\mathcal{F})$ is attained at some $x^* \in \mathrm{bd}(\mathcal{F})$, then because $\mathcal{H}$ induces non-intersecting hollows in $\mathcal{F}$, we have $x^* \in \mathcal{G}$. Hence

$$v(\mathcal{G}) \leq q(x^*) = v(\mathcal{F}) = v(\mathcal{R}(\mathcal{F})) \leq v(\mathcal{R}(\mathcal{G}))$$

as desired. So assume $v(\mathcal{F})$ is attained only at some $x^* \in \mathrm{int}(\mathcal{F})$. Then $Q \succ 0$, and x^* is the unique global minimum of $q(\cdot)$ over $\mathbb{R}^n$. If $x^* \in \mathcal{G}$ also, then a similar argument as above shows $v(\mathcal{G}) \leq v(\mathcal{R}(\mathcal{G}))$. On the other hand, if $x^* \notin \mathcal{G}$, then $x^* \in \mathrm{int}(\mathcal{E}_k)$ for some k, in which case $Q \succ 0$ implies that $v(\mathcal{G})$ is attained on $\mathrm{bd}(\mathcal{E}_k)$. Hence,

$$\begin{aligned}
v(\mathcal{G}) &= \min\{q(x) : x \in \mathrm{bd}(\mathcal{E}_k)\} \\
&= \min\{q(x) : x^T W_k x + 2w_k^T x + \omega_k = 0\} \\
&= \min\{q(x) : x^T W_k x + 2w_k^T x + \omega_k \geq 0\} \\
&= \min\{Q \bullet X + 2c^T x : W_k \bullet X + 2w_k^T x + \omega_k \geq 0,\ Y(x, X) \succeq 0\} \\
&\leq v(\mathcal{R}(\mathcal{G}))
\end{aligned}$$

where the third equality comes from $Q \succ 0$, the fourth equality comes from the fact that the Shor relaxation with one linear constraint is exact (when it is feasible and its optimal value is attained, which occurs in this case because the dual SDP is interior feasible since $Q \succ 0$) [12], and the inequality comes from the fact that $\mathcal{R}(\mathcal{G})$ is a tightening of the preceding feasible set. □

Our next theorem establishes a relationship between tight SDP relaxations and the convex hull $\mathcal{C}(\mathcal{F})$. It requires a classical separation result for nonempty closed convex sets.

Lemma 1 (Hiriart-Urruty and Lemaréchal [9])**.** *Let $K \subseteq \mathbb{R}^p$ be a nonempty, closed, and convex set, and suppose $z \notin K$. Then there exists $s \in \mathbb{R}^p$ such that $s^T z < \inf\{s^T y : y \in K\}$.*

Theorem 2. *Let $\mathcal{R}(\mathcal{F})$ be a valid SDP relaxation of $\mathcal{F}$. The equality $v(q, \mathcal{F}) = v(q, \mathcal{R}(\mathcal{F}))$ holds for all quadratic functions $q(\cdot)$ if and only if $\mathcal{R}(\mathcal{F}) = \mathcal{C}(\mathcal{F})$.*

Proof. The *if* direction follows by Proposition 1. To prove the contrapositive of the *only if* direction, first recall that $\mathcal{R}(\mathcal{F}) \supseteq \mathcal{C}(\mathcal{F})$. If there exists $(\bar{x}, \bar{X}) \in \mathcal{R}(\mathcal{F}) \setminus \mathcal{C}(\mathcal{F})$, then Lemma 1 implies the existence of $(\bar{Q}, \bar{c})$ and corresponding $\bar{q}(x) = x^T \bar{Q} x + 2\bar{c}^T x$ such that

$$v(\bar{q}, \mathcal{R}(\mathcal{F})) \leq \bar{Q} \bullet \bar{X} + 2\bar{c}^T \bar{x} < v(\bar{q}, \mathcal{C}(\mathcal{F})) = v(\bar{q}, \mathcal{F}).$$

□

An interesting application of Theorem 2 occurs in [6], where it is shown that $\mathcal{R}(\mathcal{F}) = \mathcal{C}(\mathcal{F})$ for a certain SDP relaxation $\mathcal{R}(\mathcal{F})$ when $\mathcal{F}$ corresponds to the TRS with additional nonintersecting linear constraints. This result is partially extended to the case where linear constraints are allowed to intersect on the boundary of the unit ball defining TRS in [6, Section 5], where it is argued that $v(q, \mathcal{F}) = v(q, \mathcal{R}(\mathcal{F}))$ continues to hold for any $q(\cdot)$. Applying Theorem 2, it follows that in fact $\mathcal{R}(\mathcal{F}) = \mathcal{C}(\mathcal{F})$ must also hold when the linear constraints are permitted to intersect on the boundary of the unit ball.

As a corollary of Theorems 1 and 2, we now state our second main result of the note, which gives a description of the convex hull $\mathcal{C}(\mathcal{G})$ in terms of $\mathcal{C}(\mathcal{F})$ and $\mathcal{L}(\mathcal{H})$.

Corollary 1. *If $\mathcal{H}$ induces non-intersecting hollows in $\mathcal{F}$, then $\mathcal{C}(\mathcal{G}) = \mathcal{C}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$.*

Proof. Applying Theorem 1 with $\mathcal{R}(\mathcal{F}) = \mathcal{C}(\mathcal{F})$ and $\mathcal{R}(\mathcal{G}) = \mathcal{C}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$, we see that $v(q, \mathcal{R}(\mathcal{G})) = v(q, \mathcal{G})$ for any $q(\cdot)$. Then Theorem 2 implies $\mathcal{R}(\mathcal{G}) = \mathcal{C}(\mathcal{G})$. □

We finally provide an alternative proof of Corollary 1, which connects better with existing proof techniques involving sets such as $\mathcal{C}(\mathcal{F})$ and $\mathcal{C}(\mathcal{G})$.

Proof. We prove the corollary for $l = 1$; a simple induction argument proves the result for general l. The containment $\mathcal{C}(\mathcal{G}) \subseteq \mathcal{C}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$ is easy because $\mathcal{G} \subseteq \mathcal{F}$ and $\mathcal{C}(\mathcal{G}) \subseteq \mathcal{L}(\mathcal{H})$. For the reverse containment, let (x, X) be an extreme point of $\mathcal{C}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$. If $W_1 \bullet X + 2w_1^T x + \omega_1 > 0$, then (x, X) is in fact an extreme point of $\mathcal{C}(\mathcal{F})$, i.e., $(x, X) \in \mathcal{C}(\mathcal{F})$. So assume $W_1 \bullet X + 2w_1^T x + \omega_1 = 0$, and consider the following lemma [16]:

> Let V be a symmetric matrix, and suppose $Y \succeq 0$ with $V \bullet Y = 0$ and $\mathrm{rank}(Y) = s$. Then there exists a rank-1 decomposition $Y = \sum_{p=1}^{s} y^p (y^p)^T$ such that, for all p, it holds that $y^p \neq 0$ and $(y^p)^T V y^p = 0$.

We apply this lemma with

$$V := \begin{pmatrix} \omega_1 & w_1^T \\ w_1 & W_1 \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$

and $Y := Y(x, X)$, in which case

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = Y = \sum_{p=1}^{s} (y^p)(y^p)^T = \sum_{p=1}^{s} \begin{pmatrix} x_0^p \\ x^p \end{pmatrix} \begin{pmatrix} x_0^p \\ x^p \end{pmatrix}^T$$

with each $y^p \neq 0$, $(y^p)^T V y^p = 0$, $x_0^p \in \mathbb{R}$ and $x^p \in \mathbb{R}^n$. Suppose some $x_0^p = 0$. Then $(x^p)^T W_1 x^p = 0$, which would imply $x^p = 0$ because $W_1 \succ 0$, a contradiction. Hence, in fact

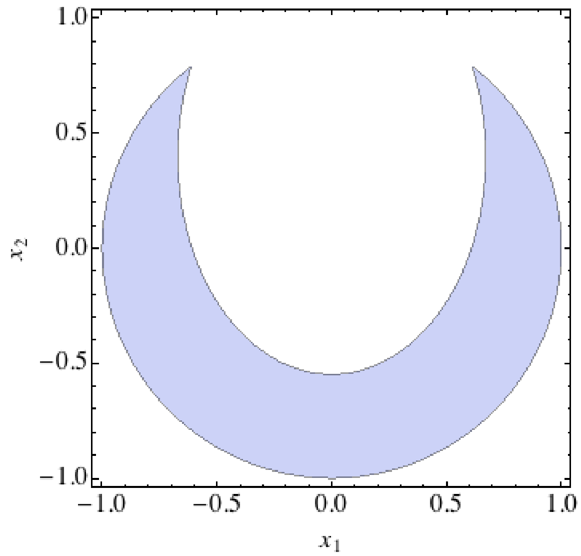

Figure 1: The feasible region $\mathcal{G}$ of Counterexample 1

each $x_0^p \neq 0$. Then defining $\bar{x}^p := x^p/x_0^p$, we have the convex combination

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} = \sum_{p=1}^{s} (x_0^p)^2 \begin{pmatrix} 1 \\ \bar{x}^p \end{pmatrix} \begin{pmatrix} 1 \\ \bar{x}^p \end{pmatrix}^T$$

where each $\bar{x}^p \in \mathrm{bd}(\mathcal{E}_1) = \mathrm{bd}(\mathcal{H}) \subseteq \mathcal{F}$. It follows that $(x, X) \in \mathcal{C}(\mathcal{F})$. $\square$

3 Counterexamples

Theorem 1 assumes that $\mathcal{H}$ induces non-intersecting hollows in $\mathcal{F}$, i.e., that each $\mathcal{E}_k \subseteq \mathcal{F}$ and all $\mathcal{E}_1, \ldots, \mathcal{E}_l$ have disjoint interiors. We now provide two examples showing that both conditions are necessary for Theorem 1, and hence also for Corollary 1.

Counterexample 1. *Let $\mathcal{F} := \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$ be the unit ball, and define $q(x) := 4x_1^2 - (x_2 + 0.5)^2$. Because of the simplicity of the quadratics involved, it is straightforward to compute $v(q, \mathcal{F}) = -2.25$ with optimal solution $x = (0, 1)$. Moreover, the SDP relaxation over $\mathcal{S}(\mathcal{F})$ is tight with $v(q, \mathcal{S}(\mathcal{F})) = -2.25$.*

Now let $\mathcal{H} := \{x \in \mathbb{R}^2 : 2x_1^2 + (x_2 - 0.4)^2 \geq 0.9\}$, and define $\mathcal{G} := \mathcal{F} \cap \mathcal{H}$, which is depicted in Figure 1. Note that the corresponding ellipsoid defined by $2x_1^2 + (x_2 - 0.4)^2 \leq 0.9$ crosses the boundary of $\mathcal{F}$. It is not difficult to check that $v(q, \mathcal{G}) = -0.25$ with optimal solution $(0, -1)$. However, the SDP relaxation over $\mathcal{S}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$ has optimal value -1.575, which shows that Theorem 1 does not hold.

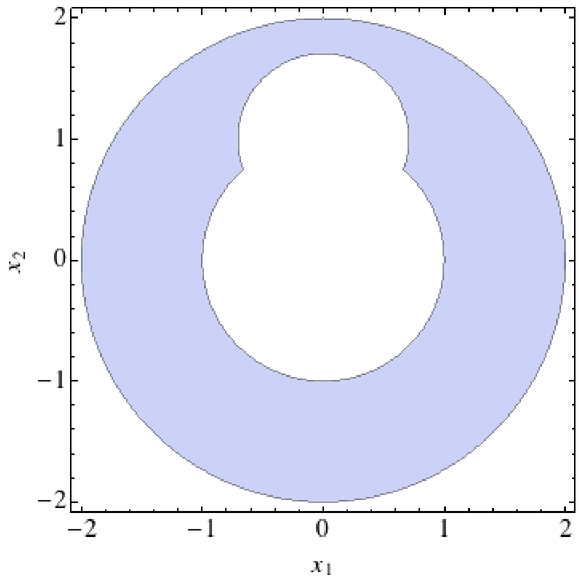

Figure 2: The feasible region $\mathcal{G}$ of Counterexample 2

It is worthwhile to note that counterexamples similar to Counterexample 1 can also be constructed for the case when an excluded ellipsoid crosses the linear portion of the boundary of a feasible set of problem TRS with an added linear inequality constraint, as discussed at the end of Section 1, e.g., for the set $\{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 1,\ x_2 \le 0.5\}$.

Counterexample 2. *Let* $\mathcal{F} := \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 4\}$*, and define* $q(x) := 2x_1^2 + (x_2 - 0.1)^2$*, which is strictly convex. One can verify that* $v(q, \mathcal{F}) = 0.81$ *with optimal solution* $x = (0, 1)$*. Moreover, the Shor relaxation* $\mathcal{S}(\mathcal{F})$ *is tight with* $v(q, \mathcal{S}(\mathcal{F})) = 0.81$*.*

Now let $\mathcal{H} := \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \ge 1, x_1^2 + (x_2 - 1)^2 \ge 0.5\}$*, and define* $\mathcal{G} := \mathcal{F} \cap \mathcal{H}$*, which is depicted in Figure 2. Clearly the two ellipsoids defining* $\mathcal{H}$ *have a nontrivial intersection. The quadratic optimal value is* $v(q, \mathcal{G}) = 1.21$ *with solution* $(0, -1)$*, but the SDP relaxation over* $\mathcal{S}(\mathcal{F}) \cap \mathcal{L}(\mathcal{H})$ *has optimal value* 0.86*, which shows that Theorem 1 does not hold.*

We remark that Counterexample 2 is written to show that Theorem 1 may fail when two excluded ellipsoids have a nontrivial intersection, but it can also be interpreted as an example where a single excluded ellipsoid intersects the boundary of a nonconvex set $\mathcal{F}$ for which $\mathcal{S}(\mathcal{F}) = \mathcal{C}(\mathcal{F})$. For this second interpretation we can start with $\mathcal{F} := \{x \in \mathbb{R}^2 : 1 \le x_1^2 + x_2^2 \le 4\}$, corresponding to a generalized TRS for which the Shor relaxation $\mathcal{S}(\mathcal{F})$ remains tight, and let $\mathcal{H} := \{x \in \mathbb{R}^2 : x_1^2 + (x_2 - 1)^2 \ge 0.5\}$.

References

[1] K. M. Anstreicher and S. Burer. Computable representations for convex hulls of low-dimensional quadratic forms. *Mathematical Programming (series B)*, 124:33–43, 2010.

[2] A. Beck and Y. C. Eldar. Strong duality in nonconvex quadratic optimization with two quadratic constraints. *SIAM J. Optim.*, 17(3):844–860 (electronic), 2006.

[3] D. Bienstock and A. Michalka. *Polynomial Solvability of Variants of the Trust-Region Subproblem*, chapter 28, pages 380–390.

[4] S. Burer. A gentle, geometric introduction to copositive optimization. *Mathematical Programming*, 151(1):89–116, 2015.

[5] S. Burer and K. M. Anstreicher. Second-order-cone constraints for extended trust-region subproblems. *SIAM Journal on Optimization*, 23(1):432–451, 2013.

[6] S. Burer and B. Yang. The trust region subproblem with non-intersecting linear constraints. *Mathematical Programming*, 149(1-2):253–264, 2015.

[7] A. Conn, N. Gould, and P. Toint. *Trust Region Methods*. Society for Industrial and Applied Mathematics, 2000.

[8] C. D'Ambrosio, A. Frangioni, L. Liberti, and A. Lodi. On interval-subgradient and no-good cuts. *Oper. Res. Lett.*, 38(5):341–345, 2010.

[9] C. L. Jean-Baptiste Hiriart-Urruty. *Fundamentals of Convex Analysis*. Springer Berlin Heidelberg, 2001.

[10] V. Jeyakumar and G. Li. Trust-region problems with linear inequality constraints: exact SDP relaxation, global optimality and robust optimization. *Mathematical Programming*, 147(1-2):171–206, 2014.

[11] Y. E. Nesterov and A. S. Nemirovskii. *Interior-Point Polynomial Algorithms in Convex Programming*. Society for Industrial and Applied Mathematics, Philadelphia, 1994.

[12] G. Pataki. On the rank of extreme matrices in semidefinite programs and the multiplicity of optimal eigenvalues. *Mathematics of Operations Research*, 23(2):pp. 339–358, 1998.

[13] T. Pong and H. Wolkowicz. The generalized trust region subproblem. *Computational Optimization and Applications*, 58(2):273–322, 2014.

[14] F. Rendl and H. Wolkowicz. A semidefinite framework for trust region subproblems with applications to large scale minimization. *Mathematical Programming*, 77(1):273–299, 1997.

[15] R. J. Stern and H. Wolkowicz. Indefinite trust region subproblems and nonsymmetric eigenvalue perturbations. *SIAM Journal on Optimization*, 5(2):286–313, 1995.

[16] J. F. Sturm and S. Zhang. On cones of nonnegative quadratic functions. *Mathematics of Operations Research*, 28(2):246–267, 2003.

[17] Y. Ye and S. Zhang. New results on quadratic minimization. *SIAM Journal on Optimization*, 14(1):245–267, 2003.